Local Properties of some Polynomial Classes of Harmonic Mappings

by

Ibrahim H. Al-Rasasi

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

MATHEMATICAL SCIENCES

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This thesis, written by IBRAHIM H. AL–RASASI under the direction of his Thesis advisor and approved by his Thesis Committee, has been presented to and accepted by the Dean of the College of Graduate Studies, in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN MATHEMATICS.

Thesis Committee

for
Dr. Abdallah K. Lyzzaik (Advisor)

Dr. Bahaettin Cengiz (Member)

Dr. Zayid Abdul–Hadi (Member)

Dr. Mohammad Al-Bar
Department Chairman

Dr. Ala H. Al-Rabeh
Dean, College of Graduate Studies

June, 1995

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THESIS ABSTRACT

FULL NAME OF STUDENT : Ibrahim H. Al–Rasasi

TITLE OF STUDY : Local Properties of Some Polynomial Classes of Harmonic Mappings

MAJOR FIELD : Mathematical Sciences

DATE OF DEGREE : June 1995

The subject of this thesis is to study the local mapping properties of the class of harmonic mappings given by

$$f(z) = h(z) + \overline{g(z)}$$

where

$$h(z) = az^{n+k-1} + bz^{n-1}, \qquad \text{and}$$
$$g(z) = cz^{n+k-1} + dz^{n-1}$$

where n and k are positive integers with $n > 1$. Here we describe and classify the critical points of f. Then we study the behaviour of f at its different kinds of critical points. Finally we describe a surface structure property for f.

MASTER OF SCIENCE DEGREE

KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

Dhahran, Saudi Arabia

June 1995

خلاصة الرسالة

اسم الطالب الكامل : ابرهيم حسين الرصاصي

عنـــوان الدراســـــة : الخصائص المحلية لأحد الرواسم الهرمونية من نوع كثيرات الحدود .

الـــتـــخـــصـــص : العلوم الرياضية.

تـــاريـــخ الشـــهـــادة : يونيو ١٩٩٥م .

موضوع هذه الرسالة هو دراسة الخصائص الهندسية المحلية لأحد أصناف الرواسم الهرمونية من نوع كثيرات الحدود ، والتي شكلها العام معطى كالآتي :

$$f(z) = h(z) + \overline{g(z)}$$

حيث

$$h(z) = az^{n+k-1} + bz^{n-1}$$

$$g(z) = cz^{n+k-1} + dz^{n-1}$$

حيث n و k اعداد صحيحة موجبة بشرط n ›١ . نبدأ بتصنيف ووصف النقاط الحرجة للراسم f ، ومن ثم نشرع في دراسة تصرف الراسم f عند نقاطه الحرجة بجميع انواعها . في الختام ، نعطي وصفاً هندسياً لتصرف الراسم f إذا عُرِّف على سطح ريماني .

درجة الماجستير في العلوم

جامعة الملك فهد للبترول و المعادن

الظهران ، المملكة العربية السعودية

يونيو ١٩٩٥م

CHAPTER - 0
INTRODUCTION

A harmonic mapping of a region G is a function of the form $f = u + iv$, where u and v are real-valued harmonic functions of G. A harmonic mapping can be written as $f = h + \bar{g}$ where h and g are analytic functions of G. Harmonic mappings have become a center of interest for complex analysts in the past decade, and interesting results, analytic and geometric, have been obtained. Recently, Lyzzaik [1] studied the general geometric properties of these mappings and concluded specific results on their local behaviour. This left the door open for applications of these results on special classes of harmonic mappings.

It is the objective of this thesis to study the geometry of the class of harmonic mappings given by

$$f(z) = h(z) + \overline{g(z)}$$

where

$$h(z) = az^{n+k-1} + bz^{n-1}$$

and

$$g(z) = cz^{n+k-1} + dz^{n-1}$$

where n and k are positive integers with $n > 1$. We study the local mapping properties of f and conclude some information about its surface structure.

The thesis is organized as follows. In chapter I, we give the general theory of light harmonic mappings. In doing so, we give the basic definitions and notations. Also, we introduce the critical points of a light harmonic mapping and state some of their properties. Then we give the necessary mappings needed to study harmonic mappings at their different kinds of critical points.

Chapters II and III from the core of the thesis. In these two chapters we start studying the local mapping properties of the above class, f, of harmonic mappings. In chapter II, we give first necessary conditions that guarantee the lightness of f and then we describe geometrically the critical points of f. This will be followed by giving some estimates about the number of cusps of f. In chapter III, we describe precisely the local behaviour of f at its different kinds of critical points.

In the last chapter, IV, we introduce the notion of a folded covering and state a theorem that tells us the fold of a general light harmonic mapping.

CHAPTER - I

LIGHT HARMONIC MAPPINGS

The subject of this chapter is to give the necessary background needed to study light harmonic mappings. In doing so, we state, in details, the basic definitions and theorems, with proofs, that are related to light harmonic mappings. In this chapter, all basic definitions, Lemmas and theorems are taken from [1, pp.135-142].

§1. Basic Definitions

Definition 1.1.1 Let W be a simply-connected domain of the complex plane $\mathcal{C}, u$ and v be real-valued harmonic functions of W. Any function of the form $f = u + iv$ is called a harmonic mapping of W.

To study harmonic mappings, first we will write them in a special form given in the following proposition.

Proposition 1.1.2 *Let $f = u + iv$ be a harmonic mapping of W. Then there exist analytic functions h and g of W such that $f = h + \bar{g}$.*

Proof: Since u and v are harmonic functions of W, they have single-valued harmonic conjugates u^* and v^* respectively so that $K = u + iu^*$ and $L = v + iv^*$ are analytic functions of W. It follows at once that

$$u = \frac{1}{2}(K + \bar{K}) \text{ and } v = \frac{1}{2}(L + \bar{L}).$$

Thus $f = \frac{1}{2}(K+\bar{K}) + i\frac{1}{2}(L+\bar{L}) = \frac{1}{2}(K+iL) + \overline{\frac{1}{2}(K-iL)} = h + \bar{g}$ where $h = \frac{1}{2}(K+iL)$ and $g = \frac{1}{2}(K-iL)$ are analytic functions of W. □

Proposition 1.1.3 *Let $f = h + \bar{g}$ be a harmonic mapping of W. Then the Jacobian of f is given by*

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

Proof. Write $h = u + iv$ and $g = u^* + iv^*$. Then $f = (u+u^*) + i(v-v^*)$ and the Jacobian of f is, by definition, given by

$$J_f = \begin{vmatrix} u_x + u_x^* & u_y + u_y^* \\ v_x - v_x^* & v_y - v_y^* \end{vmatrix}$$
$$= (u_x + u_x^*)(v_y - v_y^*) - (u_y + u_y^*)(v_x - v_x^*)$$

Since h and g are analytic, we have the Cauchy- Riemann equations:

$$u_x = v_y \ , \ u_y = -v_x \text{ and } u_x^* = v_y^* \ , \ u_y^* = -v_x^*$$

Using these Cauchy–Riemann equations, J_f simplifies as

$$\begin{aligned} J_f &= (u_x v_y - u_x^* v_y^*) - (u_y v_x - u_y^* v_x^*) \\ &= (u_x v_y - u_y v_x) - (u_x^* v_y^* - u_y^* v_x^*) \\ &= (u_x^2 + v_x^2) - (u_x^{*^2} + v_x^{*^2}) \\ &= |h'|^2 - |g'|^2. \ \square \end{aligned}$$

Definition 1.1.4 A complex-valued function f is said to be light if $f^{-1}(w)$ is either empty or totally disconnected for every $w \in C$.

Examples of light harmonic functions include the non-constant analytic or anti-analytic functions. The functions $f_1(z) = z + \bar{z}$ and $f_2 = \frac{1}{4}[z^2 - 2iz + \overline{(-2iz - z^2)}]$ are not light since f_1 maps each vertical line in C to a point on the real axis and f_2 maps the real axis to the origin.

In what follows, we will deal only with the class of all light harmonic mappings of W. This class will be denoted by $D(W)$ or simply D.

Definition 1.1.5 Let $\triangle$ be a subset of C. A function $f : \triangle \longrightarrow C$ is said to be N-valent in $\triangle$ if f admits every value in C at most N times in $\triangle$ and some value exactly N times. We say that the valency of f at $z_0 \in \triangle$ is N, written as $V_f(z_o, \triangle) = N$, if there exists a positive ϵ such that for any neighbourhood U of z_o of diameter less than ϵ, f is N-valent in U. If $\triangle$ is open, the notation $V_f(z_0)$ is used instead of $V_f(z_0, \triangle)$.

In this thesis we use $\bar{A}$, Int(A), and ∂A to denote the closure, interior, and boundary of a subset A of a topological space, and we also use I to denote the unit interval $[0, 1]$.

§2 The Critical Points of Light Harmonic Mappings

In this section, we introduce the critical points of light harmonic mappings and mention some of their properties. We first begin with the preliminary result.

Lemma 1.2.1 *Let $f = h + \bar{g}$ be a harmonic mapping of W. Then J_f is* *identical to zero in some neighbourhood if and only if there exist complex constants* $\lambda, \mu \in \mathcal{C}$, *with* $|\lambda| = 1$ *such that*

$$f(z) = \mu + 2\lambda^{-1}\, Re\ \lambda h(z) \qquad (z \in W).$$

In this case f *is not light.*

Proof. Suppose J_f is identical to zero in some local neighbourhood U of W. Then $|g'| = |h'|$ in U and hence $g' = e^{i\theta}h'$ for some $\theta \in R$, which gives $g(z) = v + e^{i\theta}h(z)$ for some $v \in \mathcal{C}$ and all $z \in U$. Since g and h are analytic functions of W, by the identity theorem we have

$$g(z) = v + e^{i\theta}h(z) \qquad (z \in W)$$

Using $f = h + \bar{g}$, we get

$$\begin{aligned} f(z) &= h(z) + \overline{v + e^{i\theta}h(z)} \\ &= \bar{v} + e^{-i\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}}h(z) + \overline{e^{i\frac{\theta}{2}}h(z)}\right] \\ &= \mu + 2\lambda^{-1}\ \mathrm{Re}\ \lambda h(z) \qquad (z \in W), \end{aligned}$$

where $\mu = \bar{v}$, $\lambda = e^{i\frac{\theta}{2}}$ and $|\lambda| = 1$.

Conversely, suppose that $f(z) = \mu + 2\lambda^{-1}\ \mathrm{Re}\ \lambda h(z)$ for all $z \in W$, where $\lambda, \mu \in \mathcal{C}$ and $|\lambda| = 1$. If $\mu = \mu_1 + i\mu_2$ and $\lambda = \lambda_1 + i\lambda_2$, then f can be rewritten as

$$f = (\mu_1 + 2\lambda_1\ \mathrm{Re}\ \lambda h) + i(\mu_2 - 2\lambda_2\ \mathrm{Re}\ \lambda h),$$

and the Jacobian of f is given by

$$J_f = \begin{vmatrix} (2\lambda_1 \text{ Re } \lambda h)_x & (2\lambda_1 \text{ Re } \lambda h)_y \\ (-2\lambda_2 \text{ Re } \lambda h)_x & (-2\lambda_2 \text{ Re } \lambda h)_y \end{vmatrix} \equiv 0$$

in W. Hence J_f is identical to zero in some local neighbourhood of W. Finally, if $f(z) = \mu + 2\lambda^{-1}$ Re $\lambda h(z)$ for all $z \in W$, then the image of W under f is a straight line, a line segment or a point; and f is not light. □

The previous lemma gives the following:

Corollary 1.2.2 *If f is light , then J_f is not identical to zero.*

Proof: If J_f is identical to zero, then $f(z) = \mu + 2\lambda^{-1}$ Re λh by the above lemma. Hence f is not light and we have a contradiction. □

The converse, however, is not true as the mapping $f_2 = \frac{1}{4}[z^2 - 2iz + \overline{(-2iz - z^2)}]$ shows. f_2 is not light since it maps the real axis to the origin.

Let $f = h + \bar{g}$ be a harmonic mapping of W. Suppose that the Jacobian of f admits zero in W but is not identical to zero. Define the quotient

$$\psi(z) = \frac{h'(z)}{g'(z)}, \; z \in W.$$

Then ψ is either a nonconstant meromorphic function or is identically constant with modulus different from unity.

Definition 1.2.3 Let f and ψ be as given above.

(i) The set of critical points of f is given by the set $J = \{z \in W : J_f(z) = 0\}$.

(ii) The set $N = \{z \in J : |\psi(z)| \neq 1\}$ is called the set of *nonfolding critical points of* f.

Lemma 1.2.4 *If* $z_0 \in N$, *then there exists a neighbourhood* U *of* z_0 *such that* $U \cap (J - \{z_0\})$ *is empty.*

Proof If $z_0 \in N$, then $J_f(z_0) = 0$ and $|\psi(z_0)| \neq 1$. This implies that $|h'(z_0)| = |g'(z_0)| = 0$ (otherwise $|\psi(z_0)| = 1$ which gives a contradiction.) It follows that $h'(z_0) = g'(z_0) = 0$ and hence z_0 is a common zero of h' and g'. Since zeros of analytic functions are isolated, there is a neighbourhood U_1 of z_0 such that $U_1 \cap N = \{z_0\}$. Also since $|\psi|$ is continuous and $|\psi(z_0)| \neq 1$, there is a neighbourhood U_2 of z_0 with the property that $|\psi(z)| \neq 1$ for all $z \in U_2$. This implies that $U_2 \cap (J \backslash N) = \phi$. It follows that if $U = U_1 \cap U_2$, then U is a neighbourhood of z_0 containing no other point of J, i.e. $U \cap J = \{z_0\}$. □

Note that $J \backslash N = \{z \in J : |\psi(z)| = 1\}$. It is easy to see that $J \backslash N$ is a set Γ_f consisting of curves which are analytic except possibly for algebraic singularities. We assume henceforth that the direction of Γ_f is that induced by ψ from the positive direction of the unit circle. It is also assumed that this direction is inherited by the Jordan subarcs of Γ_f.

Let γ be a directed Jordan subarc of Γ_f given by an analytic path $z(t), t \in I$. Then

$$\psi o z(t) = \exp\,(i\phi(t)) \;.\; t \in I$$

defines $\phi : I \rightarrow R$ as a continuously increasing function.

Before we continue, let us recall what is meant by an analytic path.

Definition 1.2.5 A continuous curve is said to be analytic if its equation can be written in the form $z = z(t), t \in$ [a, b], where

1. $z(t)$ has a power series expansion about each point in [a, b]
2. $z'(t) \neq 0$ at each point in [a, b].

Lemma 1.2.6 *With the above notation we have*

$$(f o\, z(t))' = 2\ Re\ \omega(t) \cdot\ exp\ (i\phi(t)/2)\ \ ,\ \ t \in I$$

where

$$\omega(t) = h'(z(t)) \cdot z'(t) \cdot\ exp\ (-i\phi(t)/2)$$

is an analytic path which satisfies the property that Re ω either admits zero finitely many times or is identical to zero. In the latter case, f is constant on γ and f is not light.

Proof. By writing

$$foz(t) = hoz(t) + \overline{goz(t)},$$

we have at once

$$(foz(t))' = h'(z(t)) \cdot z'(t) + \overline{g'(z(t)) \cdot z'(t)}$$

$$
\begin{aligned}
&= h'(z(t)) \cdot z'(t) + \frac{h'(z(t))}{\psi(z(t))} \cdot z'(t) \\
&= h'(z(t)) \cdot z'(t) + \overline{\exp\left(-i\phi(t)\right) \cdot h'(z(t)) \cdot z'(t)} \\
&= \exp\left(i\frac{\phi(t)}{2}\right)\left[\exp\left(-i\frac{\phi(t)}{2}\right) \cdot h'(z(t)) \cdot z'(t)\right. \\
&\quad \left. + \overline{\exp\left(-i\frac{\phi(t)}{2}\right) \cdot h'(z(t)) \cdot z'(t)}\right] \\
&= 2 \operatorname{Re} \omega(t) \cdot \exp\left(i\frac{\phi(t)}{2}\right),
\end{aligned}
$$

where $\omega(t)$ is as given in the Lemma.

Since ψ is analytic on γ, it follows that $\omega(t)$ is an analytic path. One property of analytic paths is that any pair of them either meet in a finite number of points or in an arc [4, p.243]. Applying this property to $\omega(t)$ and the imaginary axis, we conclude that either $\omega(t)$ meets the imaginary axis in a finite number of points or is a subset of the imaginary axis. This, in turn, is equivalent to the given property of Re $\omega(t)$. The remaining of the proof follows obviously. □

Lemmas 1.2.1 and 1.2.6 give at once the following result:

Theorem 1.2.7 *A harmonic mapping of W is either (a) light, (b) has a zero Jacobian, or (c) is constant on some analytic subarcs of* Γ_f.

The rest of the critical points of f are defined as follows:

Definition 1.2.8 Let $f = h + \bar{g} \in D$ and let $z_0 \in \Gamma_f$. Under the above notation of γ and ω we have:

(i) If $\psi'(z_0) \neq 0$, then z_0, with $z_0 = z(t_0), 0 < t_0 < 1$, is interior to a Jordan subarc γ of Γ_f. We call z_0 *a folding critical point of f of the*

(a) *first kind* if Re $\omega(t)$ changes signs at t_0;

(b) *second kind* if z_0 is a zero of h', or equivalently g', (which yields Re $\omega(t_0) = 0$) and Re $\omega(t)$ changes no signs at t_0.

(ii) If $\psi'(z_0) = 0$, then z_0 is called a *folding critical point of f of the third kind.*

The set of all folding critical points of f of the jth kind, $j = 1, 2, 3$, will be denoted by F_j. We set $F = \cup_{j=1}^{3} F_j$ and call F the set of folding critical points of f.

Theorem 1.2.9 *Let $f = h + \bar{g} \in D$. Then the critical points of f, folding and nonfolding, are isolated.*

Proof. Let z_0 be a critical point of f. If z_0 is nonfolding, then z_0 is isolated from the rest of the critical points of f as shown in Lemma 1.2.4. If z_0 is folding, then z_0 is isolated from the nonfolding critical points since the latter do not cluster in W. For if z_0 is not isolated from the nonfolding critical points, then every neighbourhood of z_0 contains at least one point of N. Hence the points of N cluster in W. But since the points of N are common zeros of h' and g', the identity theorem implies that h' and g' are identically zero and we have a contradiction.

Now since z_0 is a zero of finite order for $\psi(z) = \psi(z_0)$, there is a neighbourhood

Δ of z_0 which meets Γ_f in a finite number of Jordan arcs. Definition 1.2.8 states that Re $\omega(t) = 0$ at the folding critical points of the first and second kind. By Lemma 1.2.6, since Re $\omega(t)$ admits zero finitely many times, we conclude that f has a finite number of folding critical points of the first and second kinds on each subarc of Γ_f. Hence Δ can be chosen sufficiently small so that it contains z_0 as the only folding critical point. $\square$

Definition 1.2.10 A directed arc α is called

(a) convex if α is simple and the slope of its tangent is continuously increasing.

(b) locally convex at z_0 if z_0 belongs to some relatively open convex subarc β of α.

(c) locally convex if α is locally convex at every point.

(d) a harmonic cusp if there exists a parametrization $w = w(t), t \in I$ and a real t_0, $0 < t_0 < 1$, such that arg $w'(t)$ is continuously increasing function in $I \backslash \{t_0\}$ with a simple discontinuity of jump π at t_0. $w(t_0)$ is called the vertex of the cusp.

Let f be light, γ be as given above and $z_0 = z(t_0) \in$ int (γ) where $t_0 \in (t_1, t_2) \subset I$. By Lemma 1.2.6, we may assume that Re ω either

(i) changes no signs in (t_1, t_2), or

(ii) changes signs in (t_1, t_2) only once and at t_0.

From Lemma 1.2.6, we have

$$arg(foz(t))' = \frac{\phi(t)}{2} + arg(2 \text{ Re } \omega(t)).$$

If $\text{Re}\,\omega(t)$ changes no signs in (t_1, t_2), then it has a fixed argument (0 if Re $\omega(t) > 0$ and π if Re $\omega(t) < 0$). It follows that $arg(foz(t))'$ is increasing since $\phi(t)$ is increasing. If Re $\omega(t)$ changes signs in (t_1, t_2) only once and at t_0, then, as above, $arg(foz(t))'$ is increasing in (t_1, t_0) and (t_0, t_2) and $arg(foz(t))'$ has a jump of size π at t_0. Geometrically, this means that if $\alpha : z = z(t), t_1 < t < t_2$, then the arc $\beta = f(\alpha)$ is locally convex if (i) holds and is a harmonic cusp with vertex $f(z_0)$ if (ii) holds. Also it follows that f is 1-1 on every sufficiently small Jordan subarc of Γ_f.

The above discussion is summarized in the following theorem.

Theorem 1.2.11 *Let $f \in D$ and $z_0 \in \Gamma_f$. Then $f|_{\Gamma_f}$*

(a) is locally convex at every $z_0 \in \Gamma_f \backslash (F_1 \cup F_3)$,

(b) maps a neighbourhood of every $z_0 \in F_1$ to a harmonic cusp, and

(c) is locally homeomorphic at every $z_0 \notin F_3$.

§ 3 Standard Mappings and General Local Behaviour

In this section we introduce the mappings needed to study the local behaviour of $f \in D$ on Γ_f, where f and Γ_f are as given in §2. In what follows $\triangle$ denotes a Jordan domain in C and n and m denote positive integers.

Definition 1.3.1 Let f be a function of $\triangle$ containing z_0. The notation

$$f_{z_0} \sim z^n$$

means that there exist an open subset U of $\triangle$ containing z_0 and sense preserving homeomorphisms $h_1 : U \longrightarrow (|\xi| < 1)$ and $h_2 : C \longrightarrow C$ such that $h_1(z_0) = h_2 o f(z_0) = 0$ and

$$h_2 o f o h_1^{-1}(\xi) = \xi^n \qquad (|\xi| < 1). \tag{1}$$

If instead of (1), we have

$$h_2 o f o h_1^{-1}(\xi) = \bar{\xi}^n \qquad (|\xi| < 1)$$

then we write

$$f_{z_0} \sim \bar{z}^n. \tag{2}$$

Remarks 1.3.2

1. The notation $(f_{z_0} \sim z^n)$ means that $h_2 o f o h_1^{-1}$ behaves on the unit disc as does $w = z^n$, i.e., $h_2 o f o h_1^{-1}$ sends each $\frac{2\pi}{n}$– sector of the unit disc one-to-one onto the whole unit disc. For further illustration see Figure 1.1.

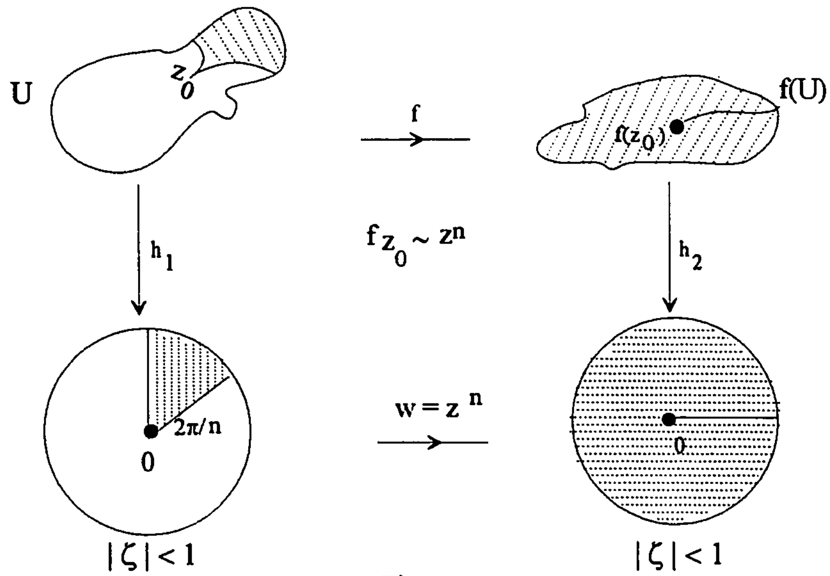

Figure 1.1

Thus if $f_{z_0} \sim z^n (f_{z_0} \sim \bar{z}^n)$, then f has the same local properties near z_0 as the mapping $w = z^n (w = \overline{z}^n)$ near the origin. This implies that n is unique and that $f_{z_0} \sim z^n$ and $f_{z_0} \sim \bar{z}^n$ cannot occur simultaneously at z_0.

2. The following two facts are well known [1, p.140].

   (a) $f_{z_0} \sim z^n (f_{z_0} \sim \bar{z}^n)$ if and only if f is open and sense preserving (reversing) in a neighbourhood of z_0.

   (b) Let $f : \Delta \longrightarrow C$ where $z_0 \in \Delta$, be a continuous function which is locally 1-1 in $\Delta - \{z_0\}$. Then there exists n such that either $f_{z_0} \sim z^n$ or $f_{z_0} \sim \bar{z}^n$.

Definition 1.3.3 Let f be a function of $\bar{\Delta}$ and let $z_0 \in \partial\, \Delta$. The notation

$$f_{z_0} \sim z^n \qquad (z \in \bar{\Delta})$$

means that there exist an open subset U of C containing z_0 and sense preserving homeomorphisms $h_1 : U \bigcap \bar{\Delta} \longrightarrow (|\xi| < 1, Im\xi \geq 0)$ and $h_2 : C \longrightarrow C$ such that $h_1(z_0) = h_2 o f(z_0) = 0$ and

$$h_2 o f o h_1^{-1}(\xi) = \xi^n \qquad (|\xi| < 1, Im\xi \geq 0). \tag{3}$$

If instead of (3) we have

$$h_2 o f o h_1^{-1}(\xi) = \bar{\xi}^n \qquad (|\xi| < 1, Im\xi \geq 0), \tag{4}$$

then we write

$$f_{z_0} \sim \bar{z}^n \qquad (z \in \bar{\Delta}).$$

Remark 1.3.4

1. An illustration of $f_{z_0} \sim z^3(z \in \bar{\Delta})$ is depicted in Figure 1.2.

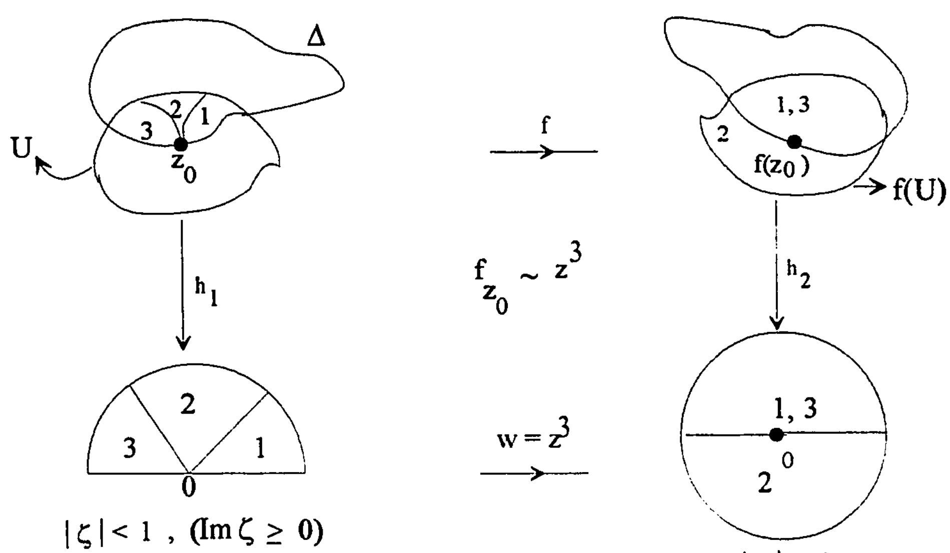

Figure 1.2

It follows that if $f_{z_0} \sim z^n (f_{z_0} \sim \bar{z}^n)$, then the local properties of f near z_0 on $\bar{\Delta} \cap U$ are similar to those of the mapping $w = z^n$ $(w = \bar{z}^n)$ near the origin on $(|\xi| < 1,\ 1m\xi \geq 0)$. Hence we conclude that n is unique and that $f_{z_0} \sim z^n$ and $f_{z_0} \sim \bar{z}^n$ cannot occur simultaneously.

2. It also follows from the definition that if $f_{z_0} \sim z^n$ or $f_{z_0} \sim \bar{z}^n$, then

   (i) f is 1-1 on the Jordan arc $U \bigcap \partial\Delta$ if n is odd, and

   (ii) f is 1-1 on each of the arc components of $(U \bigcap \partial\Delta)\backslash\{z_0\}$ if n is even. Further, f maps these arcs to the same Jordan arc.

3. If $f_{z_0} \sim z^n$ or $f_{z_0} \sim \bar{z}^n$, then f is continuous on $U \bigcap \bar{\Delta}$ and open on $U \bigcap \Delta$.

   A partial converse of this statement is given in the following:

Lemma 1.3.5 *Let γ be an open boundary arc of Δ and let $z_0 \in \gamma$. Suppose that $f : \Delta \cup \gamma \longrightarrow C$ is a continuous function which is open on Δ and 1-1 on γ. Then there exists n such that either $f_{z_0} \sim z^n$ or $f_{z_0} \sim \bar{z}^n (z \in \bar{\Delta})$,* [1, 141].

Definition 1.3.6

1. A simple arc that has one endpoint on $\partial\Delta$ and lies otherwise in Δ is called an **endcut of** Δ.

2. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be endcuts of Δ having a common endpoint $z_0 \in \Delta$ and are otherwise disjoint in $\bar{\Delta}$. Then $\bar{\Delta}$ is partitioned into m closed topological sectors which have a common vertex z_0 and which can be read as we go positively around z_0, starting from some sector, in the order $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m$. This sequence is called **a sector subdivision of** $\bar{\Delta}$ from z_0.

Definition 1.3.7 Let f be a function of $\bar{\Delta}$ and let $z_0 \in \Delta$. The notation

$$f_{z_0} \sim (z^*)^{n_1}, (z^*)^{n_2}, \ldots, (z^*)^{n_m} \tag{5}$$

where $n_1, n_2, \ldots, n_m$ are positive integers and $(z^*)^{n_k}$ equals z^{n_k} or $\bar{z}^{n_k}$ depending only on k, means that there exists a sector subdivision $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_m$ of $\bar{\Delta}$ from z_0 such that

$$f_{z_0} \sim (z^*)^{n_k} \qquad (z \in \bar{\Delta}_k) \tag{6}$$

for all k.

We close this section with a result that describes the general local behaviour of a light harmonic mapping near its folding critical points.

Theorem 1.3.8 *Let $f = h + \bar{g} \in D$ and let $z_0 \in \Gamma_f$. Then there exist positive integers $n_1, n_2, \ldots, n_{2m}$ such that*

$$f_{z_0} \sim (z^*)^{2n_1-1}, (z^*)^{2n_2-1}, \ldots (z^*)^{2n_{2m}-1}$$

where

(i) $(z^*)^{2n_k-1}$ equals z^{2n_k-1} if k is odd and $\bar{z}^{2n_k-1}$ otherwise.

(ii) $m = 1$ if $z_0 \notin F_3$

(iii) $m - 1$ is the order of z_0 as a zero of ψ' if $z_0 \in F_3$.

Proof:

(a) Let $z_0 \notin F_3$. Then $\psi'(z_0) \neq 0$ and hence z_0 is a zero of $\psi(z) = \psi(z_0)$ of order 1. Thus there exists a Jordan domain Δ containing z_0 such that Γ_f separates Δ into two Jordan domains, Δ^+ and Δ^-. By Theorems 1.2.9 and 1.2.11(c) we can choose Δ sufficiently small so that f is 1-1 on $\Delta \bigcap \Gamma_f$, $J_f > 0$ on Δ^+, and $J_f < 0$ on Δ^-. Then by the invariance of domains theorem, we can use Lemma 1.3.5 to conclude that $f_{z_0} \sim z^{2n_1-1}(z \in \bar{\Delta}^+)$ and $f_{z_0} \sim \bar{z}^{2n_2-1}(z \in \bar{\Delta}^-)$, or, $f_{z_0} \sim z^{2n_1-1}, \bar{z}^{2n_2-1}$.

(b) Let $z_0 \in F_3$ and $m-1$ be the order of z_0 as a zero of ψ'. Then z_0 is a zero of $\psi(z) = \psi(z_0)$ of order m. Let $\lambda = \psi(z_0)$ and let E be an open disc with center λ meeting the unit circle in a proper subarc, μ. Then for a sufficiently small E there exists a Jordan domain Δ containing z_0 such that $(\bar{\Delta}, \psi)$ is an m-fold covering of E with one branch point at z_0 and is of order m-1. Let the direction of μ be the positive direction of the unit circle, and let μ_1 and μ_2 be the subarcs of μ terminating and starting at λ, respectively. It follows that there exist analytic endcuts $\alpha_1, \alpha_2, \ldots, \alpha_{2m}$ of Δ with the following properties: (i) these endcuts have a common endpoint z_o and are mutually disjoint otherwise, (ii) ψ maps α_j to μ_1 if j is odd and to μ_2 if j is even, (iii) the direction of α_j is that inherited by μ via ψ, and (iv) the arcs α_j are located so that as we go positively about z_0 they can be read in the order $\alpha_1, \alpha_2, \ldots \alpha_{2m}$. This means that the arcs α_j give rise to a sector subdivision $\bar{\Delta}_1, \bar{\Delta}_2, \ldots, \bar{\Delta}_{2m}$ of $\bar{\Delta}$ from z_0, where each $\bar{\Delta}_j$ is the sector bounded by $\alpha_j, \alpha_{j+1}(\alpha_{2m+1} = \alpha_1)$ and a subarc of $\partial \Delta$. Note that

(iii) implies that α_j is directed towards (away from) z_0 if j is odd (even). This enables us to define the arc products $\gamma_j = \alpha_j\alpha_{j+1}$ and $\gamma_j = \alpha_{j+1}\alpha_j$ if j is odd and even, respectively. This implies that each γ_j is a directed Jordan arc which is analytic except at z_0 and which is mapped under ψ to μ in a direction preserving homeomorphism. Thus we can define γ_j by a path $z = z_j(t)$, $t \in I$, such that $z_0 = z(t_0)$ for some t_0, $z'(t) \neq 0$ for all $t \neq t_0$, and $\psi o z_j(t) = e^{i\phi_j(t)}$, where $\phi_j : I \longrightarrow R$ is a continuously increasing function. Now applying Lemma 1.2.6, we get

$$(foz_j(t))' = (2Re\,\omega_j(t)) \exp\,(i\phi_j(t)/2) \qquad (t \in I\backslash\{t_0\})$$

where

$$\omega_j(t) = h'(z_j(t))z_j'(t) \exp\,(-i\phi_j(t)/2).$$

Now if we choose Δ so that it contains z_0 as as the only critical point of f and Re $\omega_j(t) \neq 0$ for all $t \neq t_0$. Then Re $\omega_j(t)$ does not change sign in $[0, t_0)$ and $(t_0, 1]$. Thus Re ω_j has either the same sign in both intervals or has opposite signs. In either case $f|_{\gamma_j}$ is locally homeomorphic at z_0; in fact, $f|_{\gamma_j}$ maps a local neighbourhood of z_0 to a convex arc in the first case and to a harmonic cusp of vertex $w_0 = f(z_0)$ in the second case.

Now if f is sense preserving (reversing) in every sector $\bar{\Delta}_j$ where j is odd (even), then the result follows by the same argument of (a). $\square$

CHAPTER - II

A CLASS OF LIGHT HARMONIC MAPPINGS

The primary purpose of this thesis is to study the local properties of the class of light harmonic mappings given by

$$f(z) = h(z) + \overline{g(z)},$$

where

$$h(z) = az^{n+k-1} + bz^{n-1}$$

and

$$g(z) = cz^{n+k-1} + dz^{n-1},$$

where a, b, c and d are complex numbers and n and k are positive integers with $n > 1$.

In this chapter we start studying the above class. In §1, we give necessary conditions that guarantee the lightness of f. In §2, we describe geometrically the critical points of f. And finally, in §3, we give some estimates on the number of cusps of f.

§1 Necessary conditions for the lightness of f.

Here we give necessary conditions under which f is light. The Jacobian of f is given by

$$J_f(z) = |a(n+k-1)z^{n+k-2} + b(n-1)z^{n-2}|^2 - |c(n+k-1)z^{n+k-2} + d(n-1)z^{n-2}|^2$$

Proposition 2.1.1 *J_f is identical to zero if and only if $ad - bc = 0$ and $|a| = |c|$.*

Proof:

(i) We show first that if J_f is identical to zero then $ad - bc = 0$ and $|a| = |c|$. For this we will show the contrapositive i.e. if $ad - bc \neq 0$ or $|a| \neq |c|$, then J_f is not identical to zero.

1. If $ad - bc \neq 0$, then the linear fractional transformation $Tz = \frac{a(n+k-1)z+b(n-1)}{c(n+k-1)z+d(n-1)}$ is a Mobius transformation. Hence the set $\{z : |Tz| = 1\}$ forms a circle γ whose image under T is the unit circle. We conclude also that the set $\{z : |Tz^k| = 1\}$ forms a set of curves whose image under $w = z^k$ is γ. Since the set $\{z : |Tz^k| = 1\}$ does not form the whole plane, we conclude that J_f is not identical to zero.

2. Suppose $|a| \neq |c|$. If $ad - bc \neq 0$, then from (1) above, J_f is not identical to zero. So, suppose that $ad - bc = 0$. If $c = 0$, then $d = 0$ and f is analytic with $J_f = |h'(z)|^2$ not identical to zero. If $a = 0$, then $b = 0$ and f is anti-analytic with $J_f = |g'|^2$ not identical to zero. If $ac \neq 0$, then

$$|Tz^k| = \left| \frac{a}{c} \frac{z^k + \frac{b(n-1)}{a(n+k-1)}}{z^k + \frac{d(n-1)}{c(n+k-1)}} \right| = |\frac{a}{c}| \neq 1.$$

Hence J_f is not identical to zero.

(ii) Now we show that if $ad - bc = 0$ and $|a| = |c|$, then J_f is identical to zero. If $c = 0$, then $a = 0$ and J_f is identical to zero since $|b| = |d|$. If $c \neq 0$, then from

$$\begin{aligned} h'(z) &= a(n+k-1)z^{n+k-2} + b(n-1)z^{n-2} \text{ and} \\ g'(z) &= c(n+k-1)z^{n+k-2} + d(n-1)z^{n-2}, \\ \text{we get } ch'(z) - ag'(z) &= (bc - ad)(n-1)z^{n-2} = 0. \end{aligned}$$

This gives $h'(z) = \frac{a}{c}g'(z)$ and hence $J_f = \left|\frac{a}{c}g'\right|^2 - |g'|^2 = \left(\left|\frac{a}{c}\right|^2 - 1\right)|g'|^2 \equiv 0.$ □

Since for f to be light, it is necessary that the Jacobian of f be not identical to zero, we have the following

Corollary 2.1.2 *If f is light, then $ad - bc \neq 0$ or $|a| \neq |c|$.*

§ 2 The Critical Points of f

Proposition 2.2.1

(i) If $ad - bc \neq 0$, then

$$J = \{0\} \cup \{z : |a(n+k-1)z^k + b(n-1)| = |c(n+k-1)z^k + d(n-1)|\}.$$

(ii) If $ad - bc = 0, |a| \neq |c|$ and $ac \neq 0$, then J equals the set of zeros of h', or equivalently, the set of zeros of g'.

Proof:

(i) This follows immediately from the definition of the Jacobian.

(ii) This follows at once by noticing that the given conditions imply that $h' = \frac{a}{c}g'$ (see (ii) in the proof of proposition (2.1.1)) and hence $J_f = \left(\left|\frac{a}{c}\right|^2 - 1\right)|g'|^2$.□

Now we proceed to describe geometrically the critical points of f when $ad-bc \neq 0$ (that is, case (i) of the above proposition.) Recalling the notation introduced in chapter I, we have

$$\psi(z) = \frac{a(n+k-1)z^{n+k-2} + b(n-1)z^{n-2}}{c(n+k-1)z^{n+k-2} + d(n-1)z^{n-2}}$$

$$= \frac{a(n+k-1)z^k + b(n-1)}{c(n+k-1)z^k + d(n-1)}, \quad z \neq 0.$$

Note that we can write

$$\psi(z) = T(z^k)$$

where

$$Tz = \frac{a(n+k-1)z + b(n-1)}{c(n+k-1)z + d(n-1)}.$$

Also we have $\Gamma_f = \{z \in J : |\psi(z)| = 1\} = J \backslash \{0\}$. Since $|\psi(z)| = 1$ for every $z \in \Gamma_f$, ψ maps Γ_f into the unit circle C. Thus if $z = z(t), t \in I$, is a parametrization of Γ_f, then $\psi o z(t) = e^{i\phi(t)}$ where $\phi : I \longrightarrow R$ is a continuously increasing function.

Now we describe Γ_f. Since $\psi(z) = T(z^k), \Gamma_f$ is the preimage of the unit circle C under ψ. Since T is a Mobius transformation, the preimage of C under T is a (proper) circle if $|a| \neq |c|$ and a line otherwise (see Figure 2.1). This follows simply from the fact that $T(\infty) = \frac{a}{c}$.

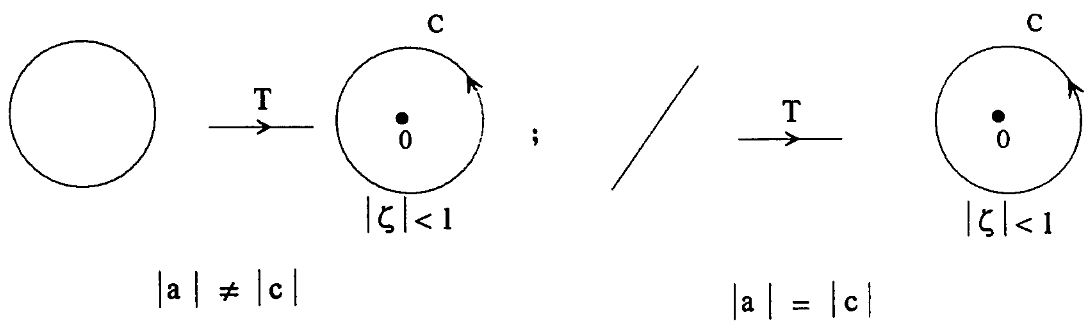

Figure 2.1

We consider these two cases separately as follows:

Case 1. $|a| \neq |c|$. Here the orientation of the circle $T^{-1}(C)$ and the relative position of the origin are of considerable concern. Noting that $T(0) = \frac{b}{d}$ and $T(\infty) = \frac{a}{c}$, we are easily led by the orientation principle to the following possibilities.

(i) If $|a| < |c|$ and $|b| < |d|$, then 0 and ∞ lies outside $T^{-1}(C)$ whose inherited orientation from C via T will be negative.

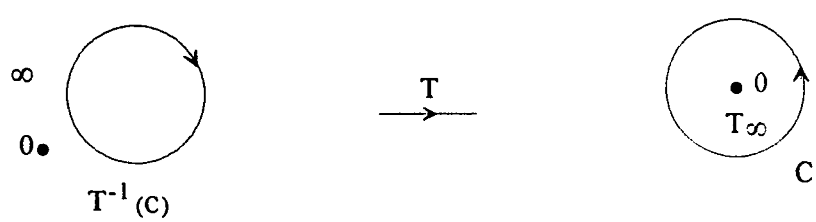

Figure 2.2

(ii) If $|a| > |c|$ and $|b| > |d|$, then 0 and ∞ lie outside $T^{-1}(C)$ whose inherited orientation from C via T will be positive.

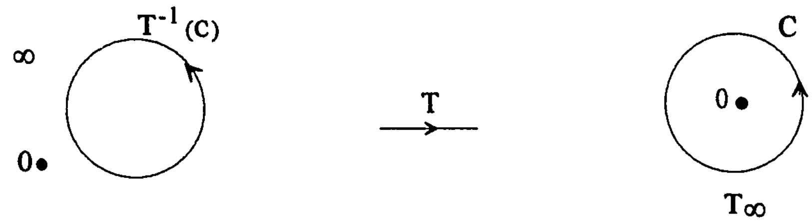

Figure 2.3

(iii) If $|a| < |c|$ and $|b| = |d|$, then 0 lies on $T^{-1}(C), \infty$ lies outside $T^{-1}(C)$ and the inherited orientation of $T^{-1}(C)$ from C via T will be negative.

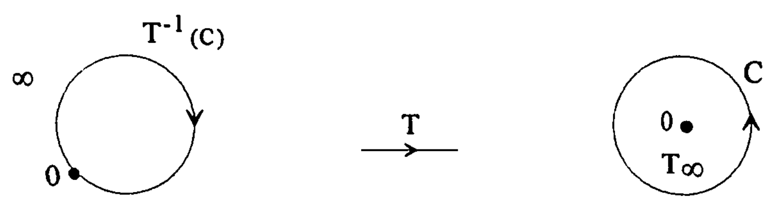

Figure 2.4

(iv) If $|a| > |c|$ and $|b| = |d|$, then 0 lies on $T^{-1}(C)$, ∞ lies outside $T^{-1}(C)$ and the inherited orientation of $T^{-1}(C)$ from C via T will be positive.

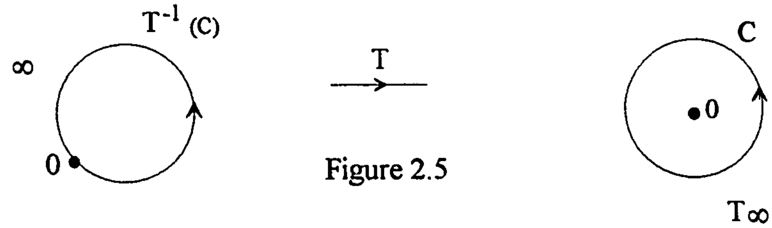

Figure 2.5

(v) If $|a| < |c|$ and $|b| > |d|$, then 0 lies inside $T^{-1}(C)$, ∞ lies outside $T^{-1}(C)$ and the inherited orientation of $T^{-1}(C)$ from C via T will be negative.

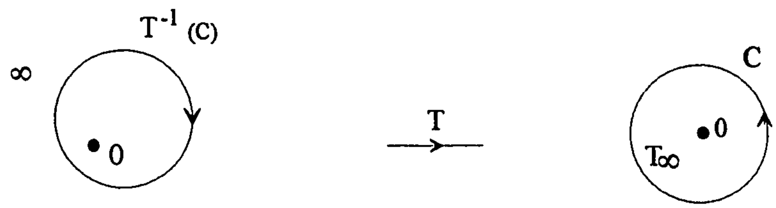

Figure 2.6

(vi) If $|a| > |c|$ and $|b| < |d|$, then 0 lies inside $T^{-1}(C), \infty$ lies outside $T^{-1}(C)$ and the inherited orientation of $T^{-1}(C)$ from C via T will be positive.

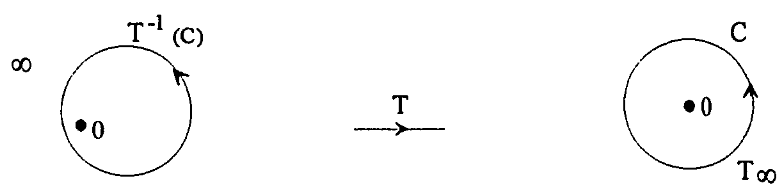

Figure 2.7

Now given $T^{-1}(C)$, the curve Γ_f is the image of $T^{-1}(C)$ under the multi-valued function $w = z^{1/k}$. At this point we need to recall a general feature of $w = z^{1/k}$, namely, that it maps the plane into k sectors each vertexed at the origin and having vertex angle of size $\dfrac{2\pi}{k}$. Regarding our situation, on each sector we will have a copy of the image of the circle $T^{-1}(C)$ under $w = z^{1/k}$. Other details will be illustrated in each case separately as shown below.

Case a: The origin is outside the circle.

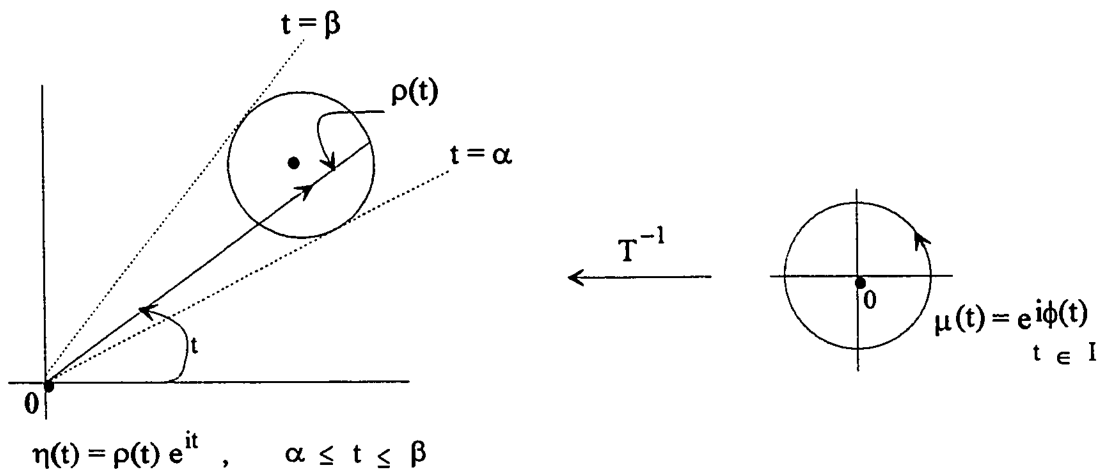

Figure 2.8

In this case $\rho(t)$ increases as one goes counterclockwise through the points whose arguments are given in the order $\alpha + \frac{\beta-\alpha}{2}$, α. $\alpha + \frac{\beta-\alpha}{2}$ and $\rho(t)$ decreases as one goes counterclockwise through the points whose arguments are given in the order $\alpha + \frac{\beta-\alpha}{2}, \beta$, and $\alpha + \frac{\beta-\alpha}{2}$.

Now the images of $\eta(t)$ under $w = z^{1/k}$ are given by

$$z_n(t) = \rho(t)^{1/k} e^{i\left(\frac{t+2\pi n}{k}\right)}, \; n = 0, 1, \ldots, k-1$$

For each $n = 0, 1, \ldots, k-1$, $z_n(t)$ gives a copy of the image of $\eta(t)$ under $w = z^{1/k}$ lying in the sector $\frac{(2n-1)\pi}{k} < t < \frac{(2n+1)\pi}{k}$. A typical image of $\eta(t)$ under $w = z^{1/k}$ is the convex Jordan arc lying in a $\frac{2\pi}{k}$ sector shown in Figure 2.9.

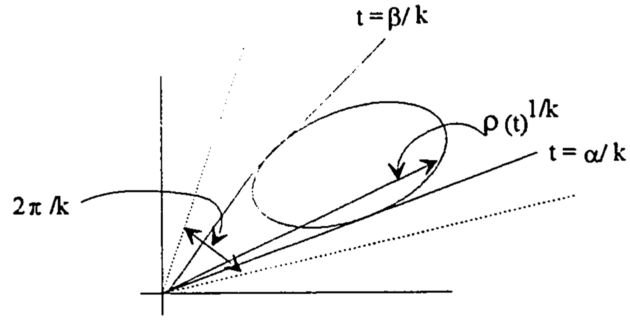

Figure 2.9

Note that $\rho(t)^{1/k}$ increases as one goes counterclockwise through the points whose arguments are given in the order $\frac{\alpha}{k}+\frac{\beta-\alpha}{2k}, \frac{\alpha}{k}, \frac{\alpha}{k}+\frac{\beta-\alpha}{2k}$ and decreases as one goes counterclockwise through the points whose arguments are given in the order $\frac{\alpha}{k}+\frac{\beta-\alpha}{2k}, \frac{\beta}{k}, \frac{\alpha}{k}+\frac{\beta-\alpha}{2k}$. Thus we deduce that Γ_f consists of k disjoint convex Jordan curves as depicted in Figure 2.10.

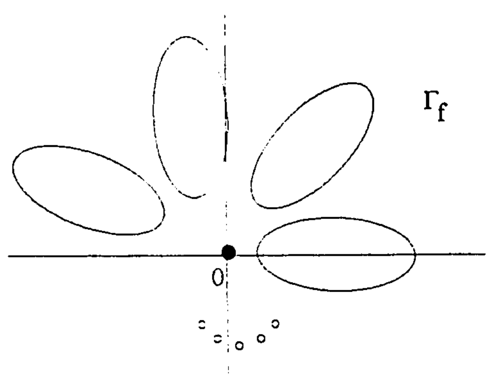

Figure 2.10

Case b: The origin is on the circle.

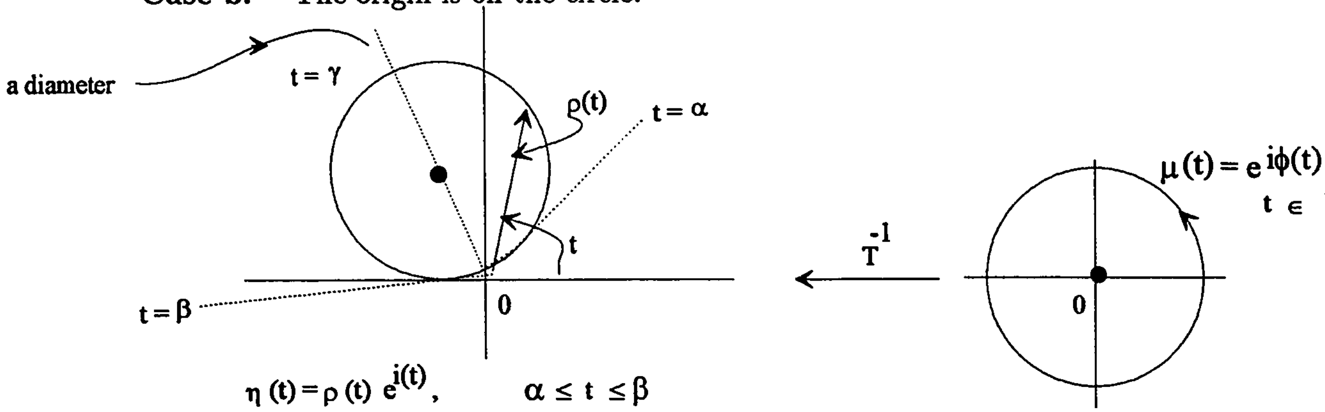

Figure 2.11

Here $\rho(t)$ increases as one goes counterclockwise through the angles α, γ and decreases as one goes counterclockwise through the angles γ, β. The images of $\eta(t)$ under $w = z^{1/k}$ are given by

$$z_n(t) = \rho(t)^{1/k}\, e^{i(\frac{t+2\pi n}{k})}\ ,\ n = 0, 1, \ldots k-1.$$

For each $n = 0, 1, \ldots, k-1$, $z_n(t)$ gives a copy of the image of $\eta(t)$ under $w = z^{1/k}$ lying in the sector $\frac{(2n-1)\pi}{k} < t < \frac{(2n+1)\pi}{k}$. A typical image of $\eta(t)$ under $w = z^{1/k}$ is given in a $\frac{2\pi}{k}$ - sector in Figure 2.12.

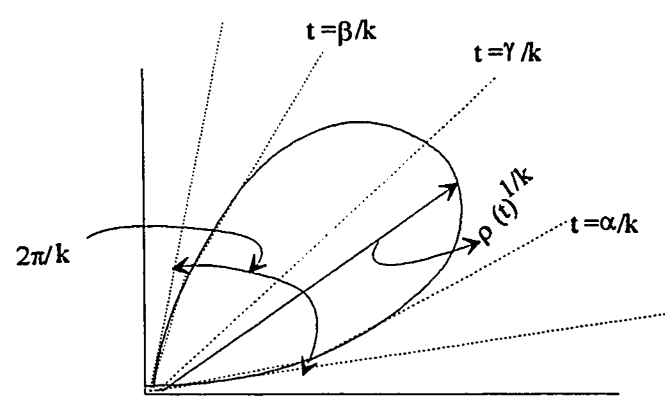

Figure 2.12

Observe that $\rho(t)^{1/k}$ increases as one goes counterclockwise through the angles $\frac{\alpha}{k}, \frac{\gamma}{k}$ and decreases as one goes counterclockwise through the angles $\frac{\gamma}{k}, \frac{\beta}{k}$. Thus we deduce that Γ_f has the shape shown in Figure 2.13.

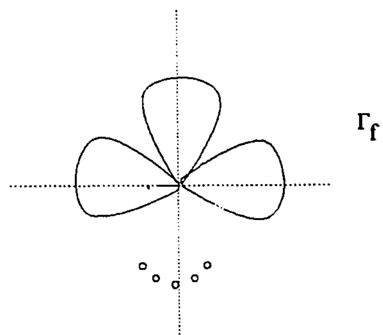

Figure 2.13

Case c: The origin is inside the circle.

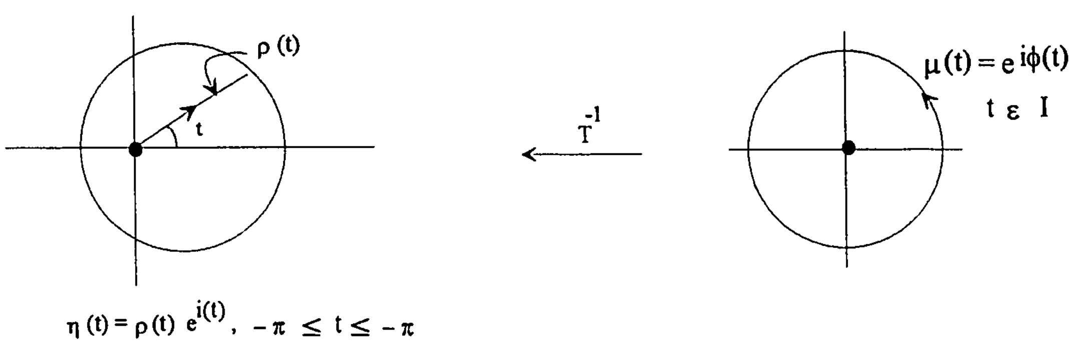

Figure 2.14

Note that $\rho(t)$ increases on $-\pi \leq t \leq 0$ and decreases on $0 \leq t \leq \pi$, and the images of $\eta(t)$ under $w = z^{1/k}$ are given by

$$z_n(t) = \rho(t)^{1/k} \, e^{i(\frac{t+2\pi n}{k})}, \quad n = 0, 1, \ldots k-1.$$

For each $n = 0, 1, \ldots, k-1$, $z_n(t)$ gives a copy of the image of $\eta(t)$ under $w = z^{1/k}$ lying in the sector $\frac{(2n-1)\pi}{k} < t < \frac{(2n+1)\pi}{k}$. The image of $\eta(t)$ given on $-\frac{\pi}{k} < t < \frac{\pi}{k}$ sector is shown in Figure 2.15.

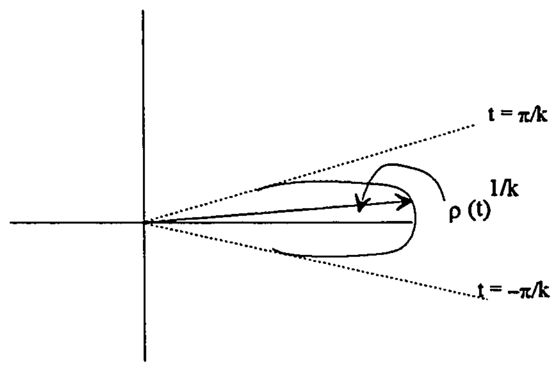

Figure 2.15

Note that $\rho(t)^{1/k}$ increases on $-\frac{\pi}{k} \le t \le 0$ and decreases on $0 \le t \le \frac{\pi}{k}$. The general shape of Γ_f is given in Figure 2.16.

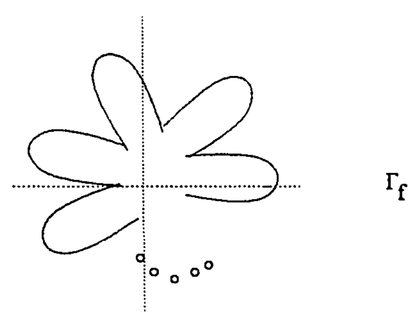

Figure 2.16

Case 2. $|a| = |c|$. In this case $T^{-1}(C)$ is a line. Γ_f is the image of $T^{-1}(C)$ under $w = z^{1/k}$. It is not easy in general to describe the image of an arbitrary line under

$w = z^{1/k}$. But fortunately we can obtain the information we want without knowing the precise image of $T^{-1}(C)$ under $w = z^{1/k}$. This will be seen in the next section.

§ 3 The Number of Cusps of f.

In this section we give some estimates about the number of cusps of f which will help to give information about the valency of f. We do this by finding an explicit equation of $T^{-1}(C)$.

Since

$$Tz = \frac{a(n+k-1)z + b(n-1)}{c(n+k-1)z + d(n-1)}$$

and $|Tz| = 1$, the equation of $T^{-1}(C)$ is given by

$$|a(n+k-1)z + b(n-1)| = |c(n+k-1)z + d(n-1)|.$$

For the moment let

$$\alpha = a(n+k-1), \beta = b(n-1), \gamma = c(n+k-1) \text{ and } \delta = d(n-1).$$

Then we have

$$|\alpha z + \beta| = |\gamma z + \delta|$$

which can be written as

$$(|\alpha|^2 - |\gamma|^2)z\bar{z} + (\alpha\bar{\beta} - \gamma\bar{\delta})z + (\bar{\alpha}\beta - \bar{\gamma}\delta)\bar{z} = |\delta|^2 - |\beta|^2. \tag{2.3.1}$$

If $|\alpha| \neq |\gamma|$, or equivalently $|a| \neq |c|$, then (2.3.1) reduces to

$$z\bar{z} + \frac{(\alpha\bar{\beta} - \gamma\bar{\delta})}{|\alpha|^2 - |\gamma|^2}z + \frac{(\bar{\alpha}\beta - \bar{\gamma}\delta)}{|\alpha|^2 - |\gamma|^2}\bar{z} = \frac{|\delta|^2 - |\beta|^2}{|\alpha|^2 - |\gamma|^2},$$

or equivalently,

$$\left(z+\frac{\bar{\alpha}\beta-\bar{\gamma}\delta}{|\alpha|^2-|\gamma|^2}\right)\left(\bar{z}+\frac{(\alpha\bar{\beta}-\gamma\bar{\delta})}{|\alpha|^2-|\gamma|^2}\right)=\frac{|\delta|^2-|\beta|^2}{|\alpha|^2-|\gamma|^2}+\frac{|\alpha\bar{\beta}-\gamma\bar{\delta}|^2}{(|\alpha|^2-|\gamma|^2)^2}$$

which simplifies to

$$\left|z+\frac{\bar{\alpha}\beta-\bar{\gamma}\delta}{|\alpha|^2-|\gamma|^2}\right|^2=\frac{|\alpha\delta-\beta\gamma|^2}{(|\alpha|^2-|\gamma|^2)^2},$$

or equivalently,

$$\left|z+\frac{\bar{\alpha}\beta-\bar{\gamma}\delta}{|\alpha|^2-|\gamma|^2}\right|=\frac{|\alpha\delta-\beta\gamma|}{||\alpha|^2-|\gamma|^2|}. \qquad (2.3.2)$$

Thus $T^{-1}(C)$ is a circle with center $P=-\frac{\bar{\alpha}\beta-\bar{\gamma}\delta}{|\alpha|^2-|\gamma|^2}$ and radius $r=\frac{|\alpha\delta-\beta\gamma|}{||\alpha|^2-|\gamma|^2|}$.

If $|\alpha|=|\gamma|$, or equivalently $|a|=|c|$, then (2.3.1) reduces to

$$(\alpha\bar{\beta}-\gamma\bar{\delta})z+(\bar{\alpha}\beta-\bar{\gamma}\delta)\bar{z}=|\delta|^2-|\beta|^2 \qquad (2.3.3)$$

and $T^{-1}(C)$ is a line.

Proposition 2.3.4 *Under the above notation and the assumption $|\alpha|\neq|\gamma|$, we have:*

(a) Suppose (i) $|\bar{\alpha}\beta-\bar{\gamma}\delta|<|\alpha\delta-\beta\gamma|$, and

(ii) $||\alpha|^2-|\gamma|^2|\,|\frac{\delta}{\gamma}|<|\alpha\delta-\beta\gamma|-|\bar{\alpha}\beta-\bar{\gamma}\delta|$.

If $|\alpha|>|\gamma|$ and $|\beta|<|\delta|$ then f has at least $2n+1$ cusps, and

if $|\alpha|<|\gamma|$ and $|\beta|>|\delta|$ then f has at least $2n-1$ cusps.

(b) Suppose (i) $|\bar{\alpha}\beta-\bar{\gamma}\delta|=|\alpha\delta-\beta\gamma|$, and

(ii) $|\frac{\delta}{\gamma}|<2\frac{|\alpha\delta-\beta\gamma|}{||\alpha|^2-|\gamma|^2|}$.

If $|\alpha| > |\gamma|$ *and* $|\beta| = |\delta|$ *then* f *has at least* $2n+1$ *cusps, and*

if $|\alpha| < |\gamma|$ *and* $|\beta| = |\delta|$ *then* f *has at least* $2n-1$ *cusps.*

(c) Suppose (i) $|\bar{\alpha}\beta - \bar{\gamma}\delta| > |\alpha\delta - \beta\gamma|$, *and*

(ii) $\left||\frac{\delta}{\gamma}| - |\frac{\bar{\alpha}\beta-\bar{\gamma}\delta}{|\alpha|^2-|\gamma|^2}|\right| < \left|\frac{\alpha\delta-\beta\gamma}{|\alpha|^2-|\gamma|^2}\right|$.

If $|\alpha| > |\gamma|$ *and* $|\beta| > |\delta|$ *then* f *has at least* $5k$ *cusps, and*

if $|\alpha| < |\gamma|$ *and* $|\beta| < |\delta|$ *then* f *has at least* $3k$ *cusps.*

In each of (a), (b) and (c), the first lower bound of cusps is attained if $\omega(t)$, $0 \le t \le 1$, *winds steadily around the origin.*

Proof: Note first that in each of (a), (b) and (c), condition (i) is necessary and sufficient for the origin to be inside, on and outside $T^{-1}(C)$, respectively, and condition (ii) is sufficient for the origin to be inside $T^{-1}(C) + \frac{\delta}{\gamma}$.

To estimate the number of cusps of f, we consider $(foz(t))'$, where $z(t)$ is a parametrization of Γ_f. A short computation shows that

$$(foz(t))' = 2e^{i\phi(t)/2} Re\,\omega(t) \qquad t \in I,$$

where

$$\omega(t) = \left(e^{i\frac{\phi(t)}{2}}\right) z'(t) z^{n-2}(t) c(n+k-1)(z^k(t) + \frac{d(n-1)}{c(n+k-1)})$$

Now the variation of arg $\omega(t)$, denoted by $\Delta arg\,\omega(t)$, is given by

$$\Delta\, arg\,\omega(t) = \Delta\,\frac{\phi(t)}{2} + \Delta arg z'(t) + (n-2)\Delta arg z(t) + \Delta arg\left(z^k(t) + \frac{d(n-1)}{c(n+k-1)}\right).$$

(a) If $|\alpha| > |\gamma|$ and $|\beta| < |\delta|$, then the orientation of Γ_f and $T^{-1}(C)+\dfrac{d(n-1)}{c(n+k-1)}$ is positive (see Figure 2.7) and

$$\begin{aligned}\Delta arg\, \omega(t) &= \frac{2\pi}{2} + 2\pi + (n-2)2\pi + 2\pi \\ &= (2n+1)\pi.\end{aligned}$$

This implies that f has at least $(2n+1)$ cusps.

If $|\alpha| < |\gamma|$ and $|\beta| > |\delta|$, then the orientation of Γ_f and $T^{-1}(C) + \frac{d(n-1)}{c(n+k-1)}$ is negative (see Figure 2.6.) and

$$\begin{aligned}\Delta arg\, \omega(t) &= \frac{2\pi}{2} - 2\pi - (n-2)2\pi - 2\pi \\ &= -(2n-1)\pi.\end{aligned}$$

This implies that f has at least $(2n-1)$ cusps.

(b) This follows exactly like (a).

(c) In this case, Γ_f consists of k disjoint Jordan curves, each lying in a $\frac{2\pi}{k}-$ sector. [see Figure 2.10.] So let $z = z(t)$ be a parametrization of one of these k Jordan curves.

If $|\alpha| > |\gamma|$ and $|\beta| > |\delta|$, then the orientation of Γ_f and $T^{-1}(C) + \frac{d(n-1)}{c(n+K-1)}$ is positive [see Figure 2.3] and

$$\Delta arg\, \omega(t) = \frac{2\pi}{2} + 2\pi + (n-2)(0) + 2\pi = 5\pi.$$

This implies that f has at least 5 cusps.

If $|\alpha| < |\gamma|$ and $|\beta| < |\delta|$, then the orientation of Γ_f and $T^{-1}(C) + \frac{d(n-1)}{c(n+k-1)}$ is negative [see Figure 2.2] and

$$\Delta arg\, \omega(t) = \frac{2\pi}{2} - 2\pi + (n-2)(0) - 2\pi = -3\pi.$$

This implies that f at least 3 cusps. Now the result follows since Γ_f consists of k disjoint Jordan curves. □

Recall that when $|\alpha| = |\gamma|$, or equivalently $|a| = |c|$, $T^{-1}(C)$ is a line whose equation is given by

$$(\alpha\bar{\beta} - \gamma\bar{\delta})z + (\bar{\alpha}\beta - \bar{\gamma}\delta)\bar{z} = |\delta|^2 - |\beta|^2.$$

[see pp.33-34]. This implies that the origin lies on the line if and only if $|b| = |d|$. The orientation of $T^{-1}(C)$ and the relative position of the origin with respect to $T^{-1}(C)$ are shown below [see Figures 2.17, 2.18, 2.19]. Recall that $T(0) = \frac{b}{d}$. We consider three cases:

(i) $|b| < |d|$

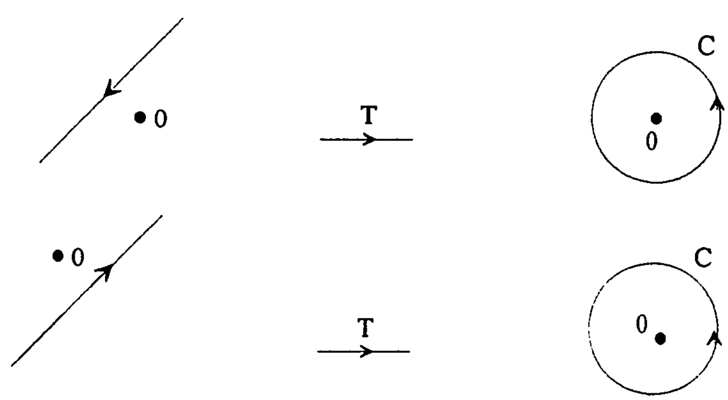

Figure 2.17

(ii) $|b| > |d|$

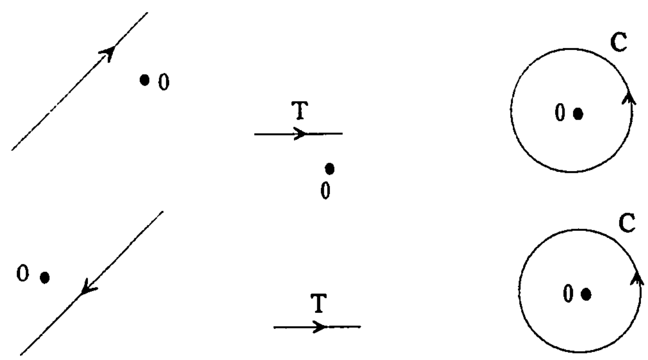

Figure 2.18

(iii) $|b| = |d|$

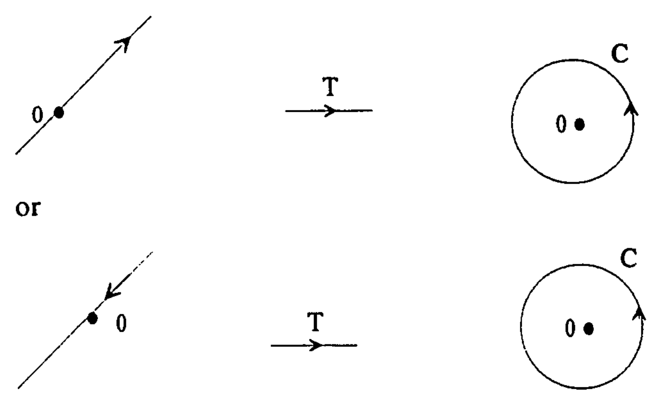

Figure 2.19

Note that the origin lies to the left (right) of the line if $|b| < |d|(|b| > |d|)$. Now Γ_f will be the image of $T^{-1}(C)$ under $w = z^{1/k}$. Fortunately, we do not need to describe $T^{-1}(C)$ precisely to find reasonable estimates for the number of cusps of f. We need only notice that

$$\Delta arg\, \gamma_n = \frac{1}{k} \Delta\ arg T^{-1}(C),$$

where γ_n is a branch of Γ_f lying in a $\frac{2\pi}{k}-$ sector. As in the proof of proposition

2.3.4, we have

$$\triangle arg\omega(t) = \triangle\frac{\phi(t)}{2} + \triangle argz'(t) + (n-2)\triangle argz(t) + \triangle arg(z^k(t) + \frac{d(n-1)}{c(n+k-1)})$$

where $z = z(t)$ is a parametrization of some γ_n, a branch of Γ_f.

If $|b| < |d|$, then $\triangle arg\,\omega(t) = \frac{2\pi}{2} + \frac{\pi}{k} + (n-2)\frac{\pi}{k} \pm \pi = (2 + \frac{n-1}{k})\pi$ or $(\frac{n-1}{k})\pi$.

If $|b| > |d|$, then $\triangle arg\,\omega(t) = \frac{2\pi}{2} - \frac{\pi}{k} - (n-2)\frac{\pi}{k} \pm \pi = (2 - \frac{n-1}{k})\pi$ or $-(\frac{n-1}{k})\pi$.

If $|b| = |d|$, then $\triangle arg\,\omega(t) = \frac{2\pi}{2} + \frac{\pi}{k} + (n-2)\frac{\pi}{k} \pm \pi = (2 + \frac{n-1}{k})\pi$ or $(\frac{n-1}{k})\pi$, or $\triangle$ $arg\,\omega(t) = \frac{2\pi}{2} - \frac{\pi}{k} - (n-2)\frac{\pi}{k} \pm \pi = (2 - \frac{n-1}{k})\pi$ or $-(\frac{n-1}{k})\pi$.

It follows that on each branch γ_n of Γ_f, f has at least one cusp. Since Γ_f consists of k branches, f has at least k cusps. Thus we have proved

Proposition 2.3.5 *Under the notation of this section, if* $|\alpha| = |\gamma|$, *or equivalently* $|a| = |c|$, *then* f *has at least* k *cusps.*

CHAPTER - III

THE LOCAL BEHAVIOUR OF f AT ITS CRITICAL POINTS

In this chapter we describe precisely the local behaviour of f at its different kinds of critical points. For easy reference, we write down the two sets of critical points of f we have already found in Proposition 2.2.1:

(i) If $ad - bc \neq 0$, then

$$J = \{0\} \cup \left\{ z : |a(n+k-1)z^k + b(n-1)| = |c(n+k-1)z^k + d(n-1)| \right\}$$

(ii) If $ad - bc = 0, |a| \neq |c|$ and $ac \neq 0$, then J equals the set of zeros of h', or equivalently, the set of zeros of g'.

§ 1. The behaviour of f at the points in N.

We begin by giving a general theorem that describes the behaviour of a general light harmonic mapping near its nonfolding critical points. Then by using this theorem, the behaviour of our class of functions, f, at the points in N will follow at once.

Theorem 3.1.1*Let $f = h + \bar{g} \in D$, $z_0 \in N$, and k and l be the order of z_0 as a zero of h' and g' respectively.*

(a) If $k < l$, then $f_{z_0} \sim z^{k+1}$

(b) If $k > l$, then $f_{z_0} \sim \bar{z}^{l+1}$

(c) If $k = l$, then $f_{z_0} \sim (z^*)^{l+1}$, where z^* equals z or $\bar{z}$ depending respectively on whether $J_f > 0$ or $J_f < 0$ in a deleted neighbourhood of z_0.

Proof. We first start with (a). The Jacobian of f can be written as

$$J_f = |h'|^2 J_1$$

where

$$J_1 = 1 - |\frac{g'}{h'}|^2$$

Since $k < l$, then $J_1(z_0) = 1 > 0$. It follows, by continuity of J_1 at z_0, that there exists a neighbourhood of z_0 on which $J_1 > 0$. We conclude that there is a Jordan domain Δ of z_0 such that $J_f > 0$ on $\Delta - \{z_0\}$ and hence f is locally 1-1 and sense preserving on $\Delta - \{z_0\}$. Since f is continuous on Δ, Remark 1.3.2 (2b) gives

$$f_{z_0} \sim z^m \qquad z \in \Delta \qquad (1)$$

for some positive integer m.

We now show that $m = k + 1$. Note that f can be written as

$$f(z) = F(z) + 2 \text{ Re } g(z) \qquad (2)$$

where $F(z) = h(z) - g(z)$. Since $k < l$, then z_0 is a zero of order k for $F' = h' - g'$ and hence z_0 is a zero of order $k+1$ for $F(z) = \xi_0$ where $\xi_0 = F(z_0)$. Thus we can find a closed Jordan domain $G \subset \Delta$ containing z_0 and a closed disk B centered at ξ_0 such that (G, F) is a $(k+1)$-sheeted covering of B. From (2), $f(\partial G)$ can be obtained from ∂B by a pointwise horizontal distortion of $k + 1$ copies of ∂B. It follows that f(∂G) meets the line $y = Imf(z_0)$ at exactly $2(k+1)$ points. But (1) implies that $f(\partial G)$ meets the line $y = Imf(z_0)$ in at least 2m points. Thus we get $2m \leq 2(k+1)$.

Now consider a sufficiently small disc B' centered at $f(z_0)$. Let $G' = f^{-1}(B')$. It follows that $f(\partial G')$ meets the line $y = Im\, f(z_0)$ at exactly 2m points. Equation(2) can be rewritten as

$$F(z) = f(z) - 2 \text{ Re } g(z) \tag{3}$$

from which we conclude that $F(\partial G')$ can be obtained by a pointwise horizontal distortion of m copies of $\partial B'$. Thus $F(\partial G')$ meets the line $y = Im\, \xi_0$ at exactly 2m points. But since F has z_0 as a zero of order $k+1, F(\partial G')$ meets the line $y = Im\, \xi_0$ at least $2(k+1)$ points. Thus we get $2m \geq 2(k+1)$. Hence $m = k+1$. This completes the proof of (a). The proofs of (b) and (c) follow in essentially the same manner. □

Now we go back to our class of harmonic mappings. Before we state the main theorem of this section, we need to mention some points.

If $ad - bc \neq 0$, then $\psi(z) = \frac{h'(z)}{g'(z)}$ has a removable singularity at $z = 0$. ψ will be analytic at $z = 0$ if we define $\psi(0) = \frac{b}{d}$. Thus if $|b| \neq |d|(|b| = |d|)$, then $z = 0 \in N(z = 0 \in F_3)$.

If $ad - bc = 0, |a| \neq |c|$ and $ac \neq 0$, then, as we have found previously, $h' = \frac{a}{c}g'$ and hence $\psi(z) = \frac{a}{c}$. Since $|a| \neq |c|$, then $|\psi(z)| \neq 1$ and in this case the critical points are nonfolding.

The above discussion together with Theorem 3.1.1 give at once the following theorem.

Theorem 3.1.2 *Let $f = h + \bar{g}$ where*

$$h(z) = az^{n+k-1} + bz^{n-1}$$

and

$$g(z) = cz^{n+k-1} + dz^{n-1}$$

(a) If $ad - bc \neq 0$ and $|b| \neq |d|$, then $z_0 = 0 \in N$ and $f_{z_0} \sim (z^*)^{n-1}$, where $z^* = z$ if $|b| > |d|$ and $z^* = \bar{z}$ if $|b| < |d|$.

(b) If $ad - bc = 0, |a| \neq |c|$ and $ac \neq 0$, then $N = J$ and

(i) If $z_0 = 0$, and $f_{z_0} \sim (z^*)^{n-1}$, where $z^* = z$ if $|a| > |c|$ and $z^* = \bar{z}$ if $|a| < |c|$.

(ii) If $z_0 \neq 0$, then $f_{z_0} \sim (z^*)^2$, where z^* is in (i).

We end this section with an interesting result whose proof uses the idea of the proof of theorem 3.1.1. It is well known by the Inverse Function Theorem that if f is a vector-valued function having $J_f(x_0) \neq 0$, then f is locally 1-1 at x_0. However, the converse is not true as shown by the function $f : R^2 \longrightarrow R^2$ given by

$$f(x, y) = \begin{bmatrix} x^3 \\ y^3 \end{bmatrix}.$$

In fact f is globally 1-1. But $J_f(0,0) = 0$. One may ask: under what conditions does the converse of the Inverse Function Theorem hold? It turns out that the converse holds if f is a harmonic mapping as the following theorem shows.

Theorem 3.1.3 *Let $f = h+\bar{g}$ be a light harmonic mapping of a simply connected domain W. If f is locally 1-1, then J_f is never zero.*

Proof. Suppose not, i.e., $J_f(z_0) = 0$ for some $z_0 \in W$. This implies that $|h'(z_0)| = |g'(z_0)|$ and hence $h'(z_0) = \lambda g'(z_0)$ for some λ with $|\lambda| = 1$. By adding and subtracting λg to the right hand side of $f = h + \bar{g}$, we get

$$f(z) = F(z) + 2\tau \text{ Re } \tau g(z) \qquad (1)$$

where $\tau^2 = \lambda$ and $F(z) = h(z) - \lambda g(z)$. Since $F'(z_0) = 0$, it follows that z_0 is a zero of F' of order $k \geq 1$. Hence z_0 is a zero of order $k \geq 2$ for $F(z) = \xi_0$ where $\xi_0 = F(z_0)$.

Since f is locally 1-1 at z_0, then there are a Jordan domain Δ of z_0 and a neighbourhood U of $f(z_0)$ such that $f : \Delta \longrightarrow U$ is 1-1. Rewrite equation (1) as

$$F(z) = f(z) - 2\tau \text{ Re } \tau g(z). \qquad (2)$$

It follows that $F(\partial\Delta)$ can be obtained by a pointwise distortion of one copy of ∂U in the direction of τ. Thus $F(\partial\Delta)$ winds once through $\partial F(\Delta)$, a contradiction since F has a zero of order $k \geq 2$ at z_0 and hence winds at least twice through $\partial F(\Delta)$. □

The above result was due to H. Lewy [5]. Lewy gave a completely different proof from the one we gave above. His proof was purely analytic.

§2 The Behaviour of f at the Points in $\Gamma_f \backslash F_3$

Here we describe the local mapping properties of f at the points in $\Gamma_f \backslash F_3$. This is given in the following

Theorem 3.2.1. *Let $f = h + \overline{g} \in D$, where*

$$h(z) = a\, z^{n+k-1} + b\, z^{n-1}$$

and

$$g(z) = c\, z^{n+k-1} + d\, z^{n-1}$$

where a, b, c and d are complex numbers and n and k are positive integers with $n > 1$, and let $z_0 \in \Gamma_f \backslash F_3$.

(a) If $z_0 \in \Gamma_f \backslash F$, then $f_{z_0} \sim z, \overline{z}$ and $V_f(z_0) = 2$.

(b) If $z_0 \in F_1$, then $f_{z_0} \sim z^3, \overline{z}$ or $f_{z_0} \sim z,\ \overline{z}^3$ and in both cases we have $V_f(z_0) =$ 3.

(c) $F_2 = \phi$.

Proof. The following proof is an adaptation of the proof of Theorem 5.1 in [1]. The proof will be given in six steps in which (a), (b) and (c) will be considered simultaneously unless otherwise stated.

STEP 1. By Theorem 1.3.8, since $z_0 \notin F_3$, it follows that there exists a Jordan domain Δ such that Γ_f separates Δ into two Jordan domains Δ^+ and Δ^- on which f is sense–preserving and sense–reversing, respectively.

Let $\gamma = \Delta \cap \Gamma_f$. By Theorem 1.2.11, we can choose Δ sufficiently small so that f maps γ homeomorphically to a Jordan arc β that is convex in (a) and a harmonic cusp with vertex $w_0 = f(z_0)$ in (b). In either case, β separates an arbitrary small open disc B centered at w_0 into two components. Using Theorem 1.3.8 again, it follows that there exists a positive integer n such that

$$f_{z_0} \sim z^{2n-1} \qquad (z \in \overline{\Delta}^+) \tag{7}$$

i.e., there exists a neighborhood N of z_0 and sense–preserving homeomorphisms $h_1 : \overline{\Delta}^+ \cap N \to (|\zeta| < 1, \mathrm{Im}\xi \geq 0)$ and $h_2 : C \to C$ such that $h_1(z_0) = h_2 \circ f(z_0) = 0$ and

$$\eta = h_2 \circ f \circ h_1^{-1}(\xi) = \xi^{2n-1} \qquad (|\xi| < 1, \mathrm{Im}\ \xi \geq 0). \tag{8}$$

Note that if we choose B such that $|h_2(w)| < 1$ for all $w \in \overline{B}$, then the set $\overline{\Delta}^+ \cap f^{-1}(\overline{B})$ consists of all $z \in \overline{\Delta}^+$ satisfying $[h_1(z)]^{2n-1} \in h_2(\overline{B})$. This set is the closure of a Jordan domain, S^+, having as a boundary arc a subarc, α, of γ that contains z_0 in its interior.

The directions of α and β will be those inherited by Γ_f via the identity map and f respectively. Let $\psi(z_0) = \lambda$ where $|\lambda| = 1$. Define the line $L : w = w_0 + t\tau \quad (t \in R)$, where τ is a chosen value of $\lambda^{1/2}$. It follows that L is tangent to β at w_0 (see Lemma 1.2.6). Denote by $[w_1, w_2]$ the diameter $B \cap L$ of B.

Suppose (a) holds. By Theorem 1.2.11, L lies to the right of β and hence the Jordan arc $h_2[w_1, w_2]$ lies in the semi–disc $\mathrm{Im}\ \eta > 0$ with the exception of the origin. Now the inverse image under f of $[w_1, w_2]$ in $\overline{S}^+$ can be easily traced by taking the inverse image of $h_2([w_1, w_2])$ under $\eta(\zeta) = \zeta^{2n-1}$ then followed by h_1^{-1}. This gives two sets of Jordan arcs $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}$ and $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}$ satisfying the properties: (i) Each α_{ij} is a cross cut of S^+ with one endpoint z_0, (ii) the arcs α_{ij} are mutually disjoint except for z_0, and (iii) f maps every α_{ij} homeomorphically to $[w_0, w_i]$.

Now suppose (b) holds. Then L crosses β at w_0. To distinguish the radii $[w_0, w_1]$ and $[w_0, w_2]$ of B, we let $[w_0, w_1]$ be to the right of β. This implies that $h_2([w_0, w_1])$ and $h_2([w_0, w_2])$ are Jordan arcs in the disc $|\eta| < 1$ which end at the origin and lie otherwise in the semi–discs Im $\eta > 0$ and Im $\eta < 0$ respectively. As above, we conclude that the inverse images under f of $[w_0, w_1]$ and $[w_0, w_2]$ form two sets of Jordan arcs which, for convenience, are also denoted by $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}$ and $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2(n-1)}$, and which satisfy the same properties of the previous sets with the exception that the set $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2(n-1)}$ is empty if $n = 1$.

STEP 2. Adding and subtracting $\lambda g(z)$ to the right hand side of $f(z) = h(z) + \overline{g(z)}$, we get

$$\begin{aligned} f(z) &= h(z) - \lambda g(z) + \overline{g(z)} + \lambda g(z) \\ &= h(z) - \lambda g(z) + \lambda^{1/2}\,[\overline{\lambda^{1/2} g(z)} + \lambda^{1/2} g(z)] \\ &= F_\lambda(z) + 2\tau \text{ Re } \tau g(z) \end{aligned} \tag{9}$$

where $F_\lambda(z) = h(z) - \lambda g(z)$ and $\tau^2 = \lambda$.

We now study the mapping properties of the analytic function F_λ at z_0. Since $\lambda = \psi(z_0) = \dfrac{h'(z_0)}{g'(z_0)}$, then $F'_\lambda(z_0) = 0$. But $F''_\lambda(z_0) \neq 0$. For if $F''_\lambda(z_0) = 0$, then $\dfrac{h''(z_0)}{g''(z_0)} = \lambda = \dfrac{h'(z_0)}{g'(z_0)}$ and hence $g'(z_0)h''(z_0) = g''(z_0)h'(z_0)$ which, after simplification, leads to $k = 0$ which contradicts $k > 0$. This means that F'_λ has a zero of order 1 at z_0. This implies that $F_2 = \phi$. For if $z_0 \in F_2$, then z_0 is a zero of both h' and g' and of the same order, say, $q \geq 1$. Consequently, F'_λ has a zero of order $q+1 \geq 2$ at z_0, and we have a contradiction.

We also conclude that z_0 is a zero of order 2 for $F_\lambda(z) = \zeta_0$ where $\zeta_0 = F_\lambda(z_0)$. It can be shown [1] that there exist a Jordan domain G containing z_0 and satisfying the properties of Δ (i.e. $J_f > 0$ on G^+ and $J_f < 0$ on G^-) and a closed disc D with center ζ_0 such that (G, F_λ) is a 2-sheeted covering of D.

Now we choose G sufficiently small so that Γ_f divides G into two components of which one, say G^+, has its closure in S^+. Note that (3) implies that $f(z)$ is a translation of $F_\lambda(z)$ in the direction of τ for all z. Since $f^{-1}|_{S^+}[w_1, w_2]$ is the union of all α_{ij}, the value of $F_\lambda(z)$, where $z \in G^+$, has the form $\zeta_0 + t\tau$ ($t \in R$) if and only if z belongs to some α_{ij}. Thus F_λ maps α_{ij} homeomorphically into one of the semi-lines $\zeta = \zeta_0 + t\tau$ and $\zeta = \zeta_0 - t\tau$ ($t \geq 0$). It follows that $\partial G^+ \backslash \gamma$ meets every α_{ij} at exactly one point a_{ij}. For if $a, b \in (\partial G^+ \backslash \gamma) \cap \alpha_{ij}$, then $F_\lambda(a) \neq F_\lambda(b)$ since F_λ is $1-1$ on α_{ij}, and $F_\lambda(a) = F_\lambda(b)$ since each of the semi-lines $\zeta = \zeta_0 + t\tau$ and $\zeta = \zeta_0 - t\tau$ ($t \geq 0$) meets ∂D at exactly one point.

We conclude that the points a_{ij} are the only points of ∂G^+ which are mapped by F_λ to the points $\zeta_1 = \zeta_0 + r\tau$ and $\zeta_2 = \zeta_0 - r\tau$, where r is the radius of D, and that the number of these points is $2n$ in (a) and $2n-1$ in (b).

STEP 3. By the choice of G, there are exactly 4 points $z_k, k = 1, \ldots, 4$, of ∂G mapped by F_λ to ζ_1 and ζ_2 and which contain the points a_{ij} as the only points z_k on ∂G^+. Let γ_1 and γ_2 be the subarcs of γ terminating and starting at z_0 respectively and let $\delta_j = F_\lambda(\gamma_j)$. The arcs δ_j are analytic (being the image of analytic arcs under an analytic function) and Jordan for sufficiently small γ. By (3), it follows that δ_1 and δ_2 lie on one side of $[\zeta_1, \zeta_2]$ if (a) holds and on different sides if (b) holds. But since F_λ is a 2–valent analytic function, it follows that $\delta_1 \backslash \{\zeta_0\}$ and $\delta_2 \backslash \{\zeta_0\}$ coincide if (a) holds and that $\delta = \delta_1 \cup \delta_2$ forms a Jordan arc if (b) holds.

Now we will find the number of points z_k in $C = \partial G^+ \backslash \gamma$. If (a) holds, then $D \backslash \delta$ consists of one component $\mathcal{U}$ with vertex angle of size 2π. Since F_λ is 2–valent, then $F_\lambda^{-1}|_G(\mathcal{U})$ consists of two components V_1 and V_2 each having a vertex angle of size π. It follows that G^+ fits exactly one component, say, V_1. Now since $\zeta_1, \zeta_2 \in \partial \mathcal{U}$, then $\partial G \cap \partial V_1$ contains two points z_k. Hence C contains exactly two points z_k.

If (b) holds, then $D \backslash \delta$ consists of two components $\mathcal{U}_1$ and $\mathcal{U}_2$ with vertex angle of size 2π and zero respectively. These two components can be viewed as topological sectors with vertex ξ_0 and arms δ_1 and δ_2. It follows that $F_\lambda^{-1}|_G(\mathcal{U}_i)$ consists of two disjoint components V_i which can also be viewed as sectors with vertex z_0. Note that the V_1 and V_2 sectors alternate around z_0 so that the sequence of sectors in G which is obtained by going positively about z_0 starts and ends with V_1 sectors or V_2 sectors. It follows that G^+ fits exactly one sector V_1 and either 0 or 2 sectors V_2. Now we have either $\zeta_1 \in \mathcal{U}_1$ and $\zeta_2 \in \mathcal{U}_2$ or $\zeta_1 \in \mathcal{U}_2$ and $\zeta_2 \in \mathcal{U}_1$. In either case, each $\partial G \cap \partial V_i$ contains exactly one point z_k. hence C contains either 1 or 3 points z_k.

STEP 4. Suppose (a) holds. Then C contains $2n$ points z_k (by Step 2) and 2 points z_k (by Step 3). This means that $2n = 2$ and hence $2n - 1 = 1$. Thus (1) becomes

$$f_{z_0} \sim z \quad (z \in \overline{\Delta^+}). \tag{10}$$

Suppose (b) holds. Then C contains $2n-1$ points z_k (by Step 2) and 1 or 3 points z_k (by Step 3). Thus we have either $2n-1=1$ or $2n-1=3$. This implies that (1) becomes

$$f_{z_0} \sim z \quad \text{or} \quad f_{z_0} \sim z^3 \qquad (z \in \overline{\Delta^+}). \tag{11}$$

STEP 5. Retracing the above steps (1 and 2) we obtain

$$f_{z_0} \sim \overline{z}^{2n-1} \qquad (z \in \overline{\Delta^-}). \tag{12}$$

From Step 3, we have seen the following:

If (a) holds, then G consists of two components V_1 and V_2 each with vertex angle of size π and G^+ fitting exactly one component V_1. We conclude that G^- fits exactly the other component V_2. Like there, it follows that $\partial G^- \backslash \gamma$ contains 2 points z_k. Thus we get $2n=2$ and hence $2n-1=1$. Thus (6) becomes

$$f_{z_0} \sim \overline{z} \qquad (z \in \overline{\Delta^-}). \tag{13}$$

If (b) holds, then G consists of 4 V_1 and V_2 sectors. Since G^+ fits exactly 1 sector V_1 and either 0 or 2 sectors V_2, then (by complementation) G^- fits either 2 or 0 sectors V_2 and exactly 1 sector V_1. Like there, it follows that $\partial G^- \backslash \gamma$ contains 3 or 1 point z_k. Thus we get either $2n-1=3$ or $2n-1=1$. Hence (6) becomes

$$f_{z_0} \sim \overline{z}^3 \quad \text{or} \quad f_{z_0} \sim \overline{z} \qquad (z \in \overline{\Delta^-}). \tag{14}$$

STEP 6. Combining Steps 4 and 5 gives the theorem except for $V_f(z_0)$ which then follows directly. $\square$

§3 The Behaviour of f at the Points in F_3

In this section we give some information about the local behaviour of f at the points in F_3. In our discussion, we will use the notation used in part (b) of the proof of Theorem 1.3.8.

Note that we can choose the paths $z_j(t)$, $\quad t \in I$, so that $z_0 = z_j(t_0)$ for all j. Now for a given j, $\mathrm{Re}\,\omega_j$ either (i) changes no signs in I or (ii) changes signs only at t_0, $0 < t_0 < 1$. It follows that $f(\gamma_j)$ is convex if (i) holds and $f(\gamma_j)$ is a harmonic cusp with vertex $w_0 = f(z_0)$ if (ii) holds. Let $\lambda = \psi(z_0)$ and define the line $L : w = w_0 + t\tau$ where $w_0 = f(z_0)$, $\ t \in R$ and τ is a value of $\lambda^{1/2}$. Then every $f(\gamma_j)$ is tangent at w_0 to L. Thus we have the following

Proposition 3.3.1. *Let $f = h + \overline{g} \in D$, where*

$$h(z) = az^{n+k-1} + bz^{n-1}$$

and

$$g(z) = cz^{n+k-1} + dz^{n-1}$$

Let $z_0 \in F_3$ and $\gamma_1, \gamma_2, \ldots, \gamma_{2m}$ be as previously defined. Then every $f(\gamma_j)$ is a convex arc or a harmonic cusp whose vertex is w_0. Furthermore, all $f(\gamma_j)$ are tangent at w_0 to the same line given above.

CHAPTER IV

LIGHT HARMONIC MAPPINGS AND FOLDED COVERINGS

In this chapter, we introduce the concept of folded coverings and establish their relation with light harmonic mappings.

§1 Some Surface Topology

We now state the necessary topology we need to study the folded coverings of a light harmonic mapping. The following definitions are taken from [3, pp. 44–45].

Definition 4.1.1. An abstract complex is a pair of finite or infinite set K and a family of finite subsets, called *simplices*, satisfying the following properties:

1. Every $\alpha \in K$ belongs to at least one and at most a finite number of simplices.

2. Every subset of a simplex is a simplex.

The dimension of a simplex is one less than the number of its elements. An n-dimensional simplex is called an n-simplex. A 0-simplex is called a vertex. The dimension of an abstract complex is the maximum dimension of its simplices. If there is no maximum, the complex is said to be of infinite dimension. In what follows, we will deal only with 2-dimensional complexes.

For an abstract complex K, there corresponds a geometric complex K_g, which is a topological space. A point in K_g is a real-valued function λ on K with the following properties:

(i) $\lambda(\alpha) \geq 0$ for all $\alpha \in K$.

(ii) The elements α with $\lambda(\alpha) > 0$ form a simplex.

(iii) $\sum_{\alpha \in K} \lambda(\alpha) = 1$.

Let S be a simplex. We obtain a corresponding subset S_g of K_g by requiring that $\lambda(\alpha) = 0$ if $\alpha \notin S$. It follows from (ii) that K_g is the union of all S_g.

Suppose S is a 2–simplex consisting of α_1, α_2 and α_3. Write $\lambda_j = \lambda(\alpha_j)$. Then we can represent a point of S_g as the triple $(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_j \geq 0$ and $\lambda_1+\lambda_2+\lambda_3 = 1$. We obtain a realization of S_g as a triangle in R^3.

Now we define a topology on K_g by stating that a subset $\mathcal{U}$ of K_g is closed if and only if $\mathcal{U} \cap S_g$ is closed in S_g for every S_g. The topology of S_g is that induced as a subspace of K_g. For further illustration, see [3, p. 45].

Definition 4.1.2. A *triangulation* of a surface F is an abstract 2–dimensional complex K and a rule σ which assigns a subset $\sigma(S)$ of F to each simplex S of K. Furthermore, σ satisfies the following conditions:

(i) $\sigma(S_1 \cap S_2) = \sigma(S_1) \cap \sigma(S_2)$.

(ii) There exists a homeomorphism of S_g onto $\sigma(S)$ which maps every $S'_g, S' \subset S$, onto $\sigma(S')$.

(iii) The union of all $\sigma(S)$ is F.

(iv) Every point on F has a neighborhood which meets only a finite number of $\sigma(S)$.

We close this section with a theorem that allows us to replace a triangulated surface by the corresponding geometric complex K_g. The proof can be found in [3, pp. 45–46].

Theorem 4.1.3. *For a given triangulation of F by a complex K, there exists a homeomorphism of K_g onto F which maps each S_g onto $\sigma(S)$.*

§2 Folded Coverings of Light Harmonic Mappings

Here we introduce the notion of a folded covering and prove that general light harmonic mappings affect folded coverings.

To motivate the concept of folded coverings, consider the map $f : (|z| < 1) \to (|w| < 1, \text{Im } w \geq 0)$ defined by $f(z) = f(\overline{z}) = z$ for Im $z \geq 0$. Then f maps the upper and lower semi-discs homeomorphically to the upper semi-disc and $f^{-1}(w)$ has two values if Im $w > 0$ and has one value if Im $w = 0$. Thus, it seems reasonable to say that f defines a covering surface of the semi-disc $(|w| < 1,\ \text{Im } w \geq 0)$ with a fold along $-1 < z < 1$.

Now we give a precise definition of the notion of folded coverings.

Definition 4.2.1. Let $\tilde{F}$ and F be Riemann surfaces and let $f : \tilde{F} \to F$. The pair $(\tilde{F}, f)$ defines a folded covering of F if there exists a triangulation of $\tilde{F}$ with complex $\tilde{K}$ such that f maps every 2–simplex of $\tilde{K}$ homeomorphically into F.

Let $\tilde{s}_1$ and $\tilde{s}_2$ be 2–simplices of $\tilde{K}$ which are adjacent along a 1–simplex $\tilde{\sigma}$. If f is sense–preserving in one and sense–reversing in the other, then we call $\tilde{\sigma}$ a part of the fold of the covering, and we call the union of all such $\tilde{\sigma}$ the fold of the covering [1, p. 151].

To illustrate the definition, let $\tilde{s}_1, \tilde{s}_2$ and $\tilde{\sigma}$ be as given in the definition. Let $p_0 \in$ Int $\tilde{\sigma}$, $q_0 = f(p_0)$ and $\sigma = f(\tilde{\sigma})$. Then $p_0 \notin \partial(\tilde{s}_1 \cup \tilde{s}_2)$ and there exists a closed Jordan domain Δ, with $q_0 \in$ Int Δ, which is divided by σ into two components whose closures are denoted by Δ^+ and Δ^-. Let $\tilde{\Delta} = f^{-1}|_{\tilde{s}_1 \cup \tilde{s}_2}(\Delta)$. Then $\tilde{\Delta}$ is a closed Jordan domain with $p_0 \in$ Int $\tilde{\Delta}$ and is divided by $\tilde{\sigma}$ into two components whose closures $\tilde{\Delta}^+$ and $\tilde{\Delta}^-$ satisfy $\tilde{\Delta}^+ \subset \tilde{s}_1$ and $\tilde{\Delta}^- \subset \tilde{s}_2$.

There exist sense-preserving homeomorphisms $h_1 : \tilde{\Delta} \to (|\zeta| < 1)$ and $h_2 : \Delta \to (|\eta| < 1)$ such that $h_1(p_0) = h_2(q_0) = 0$ and $h_1(p)$ and $h_2(q)$ are real if $p \in \tilde{\sigma}$ and $q \in \sigma$ respectively (see Figure 3.1). It follows that the closed semi-discs Im $\zeta \geq 0$ and Im $\zeta \leq 0$ are mapped under $h_1^{-1} \circ f \circ h_2$ to one of the closed semi-discs Im $\eta \geq 0$ or Im $\eta \leq 0$. Let us assume that $h_1^{-1} \circ f \circ h_2$ is a homeomorphism between the closed semi-discs Im $\zeta \geq 0$ and Im $\eta \geq 0$. Now f is either (i) sense-preserving in $\tilde{s}_1$ and $\tilde{s}_2$, or (ii) sense-preserving in $\tilde{s}_1$ and sense-reversing in $\tilde{s}_2$. The first case implies that $h_1^{-1} \circ f \circ h_2$ is a homeomorphism between the discs $|\zeta| \leq 1$ and $|\eta| \leq 1$, while the second implies that $h_1^{-1} \circ f \circ h_2$ is a homeomorphism between each of the closed semi-discs Im $\zeta \geq 0$ and Im $\zeta \leq 0$ and the semi-disc Im $\eta \geq 0$. Thus f maps $\tilde{\Delta}$ homeomorphically to Δ in the first case and f maps both of $\tilde{\Delta}^+$ and $\tilde{\Delta}^-$ homeomorphically to either Δ^+ or Δ^- in the second case. This justifies the notion of a fold of covering, see Figure 3.1.

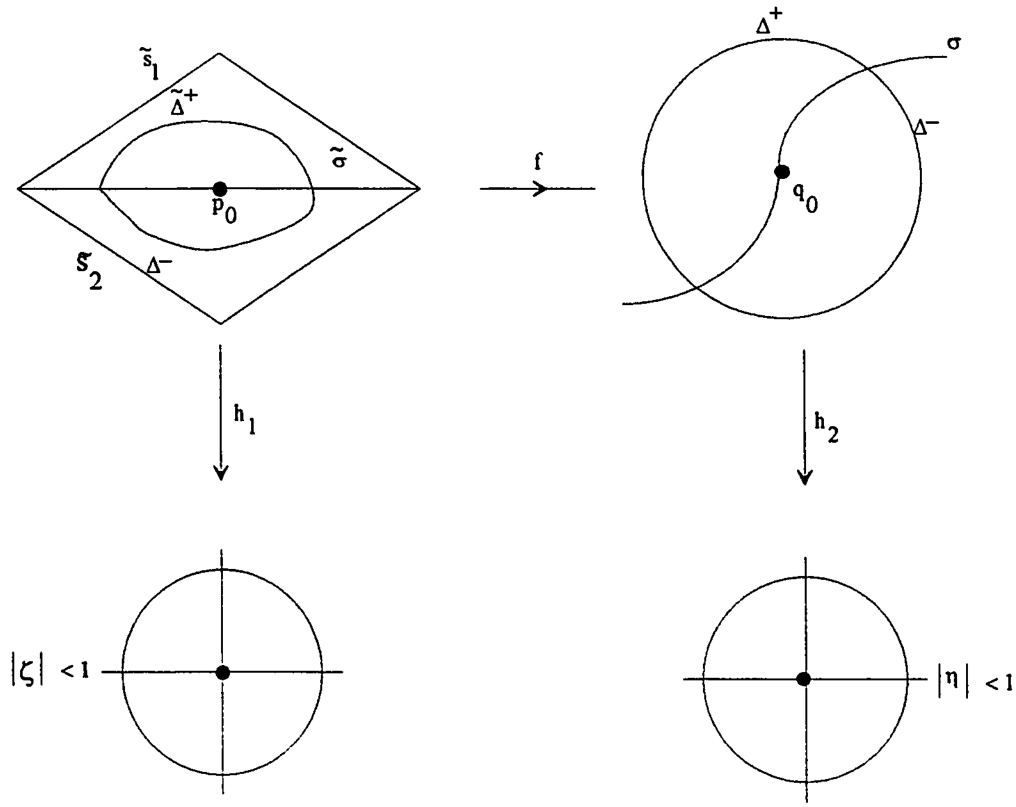

Figure 3.1

In chapter 3, we described the local mapping properties of a certain class of light harmonic mappings defined on a simply connected domain W. We found that these properties are of purely local nature. This enables us to extend the notions and results of the previous chapters to light harmonic mappings defined on Riemann surfaces.

In fact we now give a general property of a general light harmonic mapping defined on a Riemann surface. Suppose that (W, ϕ) is a Riemann surface with conformal structure ϕ, and $f : W \to C$ is a light harmonic mapping of W. Note that we can assume the associated sets $\Gamma_f, N, F = \bigcup_{j=1}^{3} F_j$ and the local behaviour of f at the points of these sets.

Now we state the main theorem of this section. The proof can be found in [1].

Theorem 4.2.2. *Let f be a light harmonic mapping of (W, ϕ). Then (W, f) is a folded covering of C whose fold is Γ_f.*

§3 A Concluding Remark

We close the thesis with a remark. Using the number of critical points, folding and nonfolding, of a folded covering, together with the degree theorem, one can conclude results about the global valency of these coverings. Applying these to special harmonic mappings, we can draw conclusions about their valencies. This would be a matter of interest in a forthcoming work.

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